

Quasisymmetric parametrizations of two-dimensional metric planes

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ABSTRACT

The classical uniformization theorem states that any simply connected Riemann surface is conformally equivalent to the disk, the plane, or the sphere, each equipped with a standard conformal structure. We give a similar uniformization for Ahlfors 2-regular, linearly locally connected metric planes; instead of conformal equivalence, we are concerned with quasisymmetric equivalence.

1. Introduction

Quasisymmetric maps are a generalization of conformal mappings of Euclidean space to the metric space setting. Analogous to the uniformization theorem for Riemann surfaces, the task of characterizing a given metric space up to quasisymmetry is of general interest. The spaces \mathbb{R}^n and \mathbb{S}^n equipped with the standard metric are of particular interest in this problem, partially because a self-homeomorphism of Euclidean space is quasisymmetric if and only if it is quasiconformal. As a result, the theory of quasiconformal mappings provides a guiding light. Tukia and Väisälä [19] gave a simple intrinsic characterization of metric spaces quasisymmetrically equivalent to \mathbb{S}^1 : a metric space X homeomorphic to \mathbb{S}^1 is quasisymmetrically equivalent to it if and only if X is doubling and linearly locally connected (LLC). A similar characterization was also given for \mathbb{R} . For $n \geq 3$, a complete characterization of \mathbb{S}^n and \mathbb{R}^n has yet to be given, and examples of Semmes [15] have shown that the problem is exceedingly difficult.

In this paper, we focus on the case $n = 2$. Bonk and Kleiner [4] found necessary and sufficient conditions for a metric space to be quasisymmetrically equivalent to \mathbb{S}^2 . Under the additional assumption of Ahlfors 2-regularity, this characterization is the same as in the one-dimensional case.

THEOREM 1.1 (Bonk–Kleiner). *Let X be an Ahlfors 2-regular metric space homeomorphic to \mathbb{S}^2 . Then X is quasisymmetrically equivalent to \mathbb{S}^2 if and only if X is linearly locally connected.*

The purpose of this paper is to extend this result to metric spaces homeomorphic to the plane. Throughout, we will use \mathbb{S}^2 , \mathbb{S}^{2*} , \mathbb{D}^2 , \mathbb{R}^2 , and \mathbb{R}_+^2 to denote the sphere, the once-punctured sphere, the open unit disk, the plane, and the open half-plane, respectively, each endowed with the metric inherited from the ambient Euclidean metric. We will denote the completion of a metric space X by \bar{X} , and the metric boundary by $\partial X := \bar{X} - X$. Our main result is the following theorem.

Received 11 September 2006; revised 22 June 2007; published online 2 May 2008.

2000 *Mathematics Subject Classification* 30C65.

The author was partially supported by NSF grants DMS-0200566, DMS-0244421, and RTG- 0602191.

THEOREM 1.2. *Let X be an Ahlfors 2-regular and linearly locally connected metric space homeomorphic to the plane or to the sphere.*

- (i) *If X is bounded and complete, then X is quasimetrically equivalent to \mathbb{S}^2 .*
- (ii) *If X is bounded and $\text{card}(\partial X) = 1$, then X is quasimetrically equivalent to \mathbb{S}^{2*} .*
- (iii) *If X is bounded and $\text{card}(\partial X) \geq 2$, then X is quasimetrically equivalent to \mathbb{D}^2 .*
- (iv) *If X is unbounded and complete, then X is quasimetrically equivalent to \mathbb{R}^2 .*
- (v) *If X is unbounded and not complete, then X is quasimetrically equivalent to \mathbb{R}_+^2 .*

The statements (i), (ii), (iv), and (v) are quantitative in the sense that the distortion function of each quasimetricity can be chosen to depend only on the constants associated to the Ahlfors 2-regularity and linear local connectedness conditions. In statement (iii), the distortion function also depends on the ratio $\text{diam } X / \text{diam } \partial X$.

Conversely, if X is a metric space that is quasimetrically equivalent to any of \mathbb{S}^2 , \mathbb{S}^{2*} , \mathbb{D}^2 , \mathbb{R}^2 , and \mathbb{R}_+^2 , then X is linearly locally connected with constant depending only on the distortion function of the quasimetricity.

Theorem 1.2 shows that in order to determine the quasimetricity type of an Ahlfors 2-regular, linearly locally connected metric space homeomorphic to the plane, we only need to know if it is bounded, and (roughly) how many non-convergent Cauchy sequences exist. As quasimetric homeomorphisms map bounded sets to bounded sets and Cauchy sequences to Cauchy sequences, this is in the minimal information required to make such a determination. Example 5.3 below shows that the dependence of the distortion function of the quasimetricity in Theorem 1.2(iii) on the ratio $\text{diam } \partial X / \text{diam } X$ cannot be avoided. The final statement of Theorem 1.2 is well known and is discussed in Remark 2.5 below.

Theorem 1.1 has an interesting application to hyperbolic geometry. A well-known conjecture of Cannon states that for every Gromov hyperbolic group G with boundary at infinity $\partial_\infty G$ homeomorphic to \mathbb{S}^2 , there exists a discrete, co-compact, and isometric action of G on hyperbolic 3-space. By the work of Sullivan [17] and Tukia [18], this conjecture is equivalent to the following statement: if G is a Gromov hyperbolic group, then $\partial_\infty G$ is homeomorphic to \mathbb{S}^2 if and only if $\partial_\infty G$ is quasimetrically equivalent to \mathbb{S}^2 . The boundary $\partial_\infty G$ of a Gromov hyperbolic group has a natural family of LLC and Ahlfors regular metrics. Thus, Theorem 1.1 confirms Cannon's conjecture if one of these metrics is Ahlfors 2-regular. Since this is not always the case, it is of particular interest to relax the Ahlfors regularity assumptions in Theorems 1.1 and 1.2. Recent progress on this problem includes [5] and [6].

Theorem 1.1 is quantitative in the same sense as Theorem 1.2. Theorem 1.2(i) is merely a rephrasing of Theorem 1.1, included for completeness of the statement. The authors of [4] note that the methods used to prove Theorem 1.1 can also be used to establish Theorem 1.2(iv). However, this approach requires the use of technical tools such as K -approximations of metric spaces and a discrete modulus, and has not been carried out in detail. The methods employed in this paper are substantially more elementary, provided that one accepts Theorem 1.1.

An outline of the proof of Theorem 1.2 is as follows. Let X be as in the hypotheses of Theorem 1.2, and suppose that X is a bounded space. Bounded and Ahlfors regular spaces are totally bounded. Thus, if X is complete, then it is homeomorphic to \mathbb{S}^2 . Theorem 1.1 then applies, proving Theorem 1.2(i). If $\text{card}(\partial X) = 1$, then X is homeomorphic to the plane. Furthermore, \bar{X} is homeomorphic to the one-point compactification of X , which is \mathbb{S}^2 . Applying Theorem 1.1 produces a quasimetric equivalence of \bar{X} and \mathbb{S}^2 , which restricts to a quasimetric equivalence of X and \mathbb{S}^{2*} .

If $\text{card}(\partial X) \geq 2$, we show that ∂X is homeomorphic to a circle. This step is the core of the paper, and is a consequence of the following more powerful theorem.

THEOREM 1.3. *Let X be a λ -LLC metric space homeomorphic to the disk. If \bar{X} is compact and ∂X contains at least two points, then ∂X is homeomorphic to the circle \mathbb{S}^1 and is λ' -LLC, where λ' depends only on λ . If, in addition, ∂X is doubling then it is quasimetrically equivalent to \mathbb{S}^1 , and the distortion function of the quasimetric can be chosen to depend only on λ and the doubling constant.*

Note that if X is doubling, then ∂X is doubling as well. To prove Theorem 1.3, we study the delicate interaction between the topological and metric properties of X . We show that ∂X is a locally connected metric continuum such that the removal of any one point does not disconnect the space, while the removal of any two does. A theorem of point-set topology states that such a space is homeomorphic to the circle [21]. In fact, our proof is quantitative, which leads to the additional conclusions regarding the LLC condition and quasimetricity.

Once it is established that ∂X is homeomorphic to the circle, we may isometrically embed X into the ‘doubled’ space X' which is obtained by gluing two copies of \bar{X} together along ∂X . The space X' is homeomorphic to \mathbb{S}^2 and is again Ahlfors 2-regular and LLC, and so we may apply Theorem 1.1 to it. The image of X under the resulting quasimetricity is an LLC domain in \mathbb{S}^2 with boundary homeomorphic to the circle. It is well known that such a domain is quasimetrically equivalent to \mathbb{D}^2 (see Theorem 2.6 below). Composing the various quasimetricities yields the desired result.

In the case where X is unbounded, we construct a new metric on X which results in a bounded metric space \hat{X} , which we call the ‘warp’ of X . This process, also employed in [5], is analogous to obtaining the standard (extrinsic) metric on \mathbb{S}^{2*} from the standard metric on \mathbb{R}^2 via stereographic projection. Similar warping processes for length spaces have recently been examined by Balogh and Buckley [2]. We show that \hat{X} is again Ahlfors 2-regular and LLC, and that the boundary of \hat{X} can be identified with $\partial X \cup \{\infty\}$. Applying the bounded cases discussed above to \hat{X} provides a quasimetricity $\hat{f}: \hat{X} \rightarrow Y$, where Y is either \mathbb{S}^{2*} or \mathbb{D}^2 . The warping process is designed such that the identity map $X \rightarrow \hat{X}$ is quasi-Möbius. This implies that \hat{f} descends to a quasimetricity $f: X \rightarrow Z$, where Z is \mathbb{R}^2 or \mathbb{R}_+^2 .

Theorem 1.2(iv) has already been used in [3] to relate the quasiconformal Jacobian problem to the classification of bi-Lipschitz images of the plane. The quasiconformal Jacobian problem in the plane asks which non-negative locally integrable functions (weights) on \mathbb{R}^2 are comparable to the Jacobian of a quasiconformal homeomorphism of \mathbb{R}^2 . Suppose that given any weight, one can determine whether it is comparable to a quasiconformal Jacobian. Then one can also determine whether a given metric space is bi-Lipschitz equivalent to the plane. Let (X, d) be a metric space. If X is bi-Lipschitz equivalent to the plane, then it is homeomorphic to the plane, Ahlfors 2-regular, LLC, unbounded, and complete. Thus we may assume that X satisfies the hypotheses of Theorem 1.2(iv). Let $f: X \rightarrow \mathbb{R}^2$ be the resulting quasimetric homeomorphism. Denote the Lebesgue measure on \mathbb{R}^2 by m_2 , and the 2-dimensional Hausdorff measure on X by \mathcal{H}^2 . Elementary properties of quasimetric maps show that the pushforward measure $\mu = f_*\mathcal{H}^2$ is a metric doubling measure, and so a theorem of David and Semmes [7] implies that μ satisfies

$$d\mu(x) = w(x)dm_2(x)$$

for a so-called strong A_∞ -weight w on \mathbb{R}^2 . It is shown in [3] that X is bi-Lipschitz equivalent to \mathbb{R}^2 if and only if w is comparable to the Jacobian of a quasiconformal homeomorphism of the plane.

As mentioned above, Theorem 1.2 can be viewed as a generalization of the classical uniformization theorem for Riemann surfaces. One might also ask if other uniformization theorems can be similarly generalized. It seems that techniques similar to those in this paper might be used to prove the following version of Koebe’s uniformization on to circle domains: if

X is a bounded, Ahlfors 2-regular, and LLC metric space homeomorphic to a domain in \mathbb{S}^2 with n boundary components, then X is quasisymmetrically equivalent to $\mathbb{S}^2 - \bigcup_{i=1}^n D_i$, where $\{D_i\}$ is a pairwise disjoint collection of closed balls or points. In light of the work of He and Schramm [10], one might also ask if such a theorem exists when countably many boundary components are allowed. Bonk, in his forthcoming paper ‘Uniformization of Sierpinski carpets in the plane’, has recently given a result in this direction in the context of Sierpinski carpets.

The techniques in this paper can also be used to show a local version of Theorem 1.2. Let X be a proper and locally Ahlfors 2-regular metric space which is homeomorphic to a surface. Assume furthermore that X is linearly locally contractible on compacta; that is, that for every compact $K \subseteq X$ there is a constant Λ such that every ball $B(x, r)$ with $x \in K$ and $0 < r \leq \Lambda^{-1}$ is contractible inside of $B(x, \Lambda r)$. Then for each point $x \in X$, there is a neighborhood of x which is quasisymmetrically equivalent to \mathbb{D}^2 . This statement plays a role in the program of Heinonen *et al.* [12] in determining which submanifolds of \mathbb{R}^n are locally bi-Lipschitz equivalent to \mathbb{R}^2 . See [22].

2. Notation, definitions, and preliminary results

Where it will not cause confusion, we will refer to a metric space (X, d) by X . For $a \in X$ and $r > 0$, we will use the following notation:

$$B_{X,d}(a, r) := \{x \in X : d(a, x) < r\},$$

$$\bar{B}_{X,d}(a, r) := \{x \in X : d(a, x) \leq r\}.$$

If $U \subseteq X$ and $\epsilon > 0$, then we denote the ϵ -neighborhood of U in X by $\mathcal{N}_\epsilon^{X,d}(U)$. We will often use $B(a, r)$, $B_d(a, r)$, or $B_X(a, r)$ in place of $B_{X,d}(a, r)$. A similar convention will be used for closed balls, neighborhoods, and other objects which depend on the space (X, d) .

Let (X, d_X) and (Y, d_Y) be metric spaces. A homeomorphism $f : X \rightarrow Y$ is called quasisymmetric if there exists a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that for all triples $a, b, c \in X$ of distinct points, we have

$$\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))} \leq \eta \left(\frac{d_X(a, b)}{d_X(a, c)} \right).$$

We will call the function η a distortion function of f ; when η needs to be emphasized, we say that f is η -quasisymmetric. If f is a quasisymmetric homeomorphism, then f^{-1} is as well. Thus we say that metric spaces X and Y are quasisymmetric or quasisymmetrically equivalent if there is a quasisymmetric homeomorphism from X to Y . We summarize some basic properties of quasisymmetric mappings in the following proposition. Proofs can be found in [11, Chapter 10; 19].

PROPOSITION 2.1. *Let $f : X \rightarrow Y$ be an η -quasisymmetric homeomorphism of metric spaces. Then we have the following.*

- (i) *If $g : Y \rightarrow Z$ is a θ -quasisymmetric homeomorphism, then $g \circ f$ is a $\theta \circ \eta$ -quasisymmetric homeomorphism.*
- (ii) *If $A \subseteq B \subseteq X$ are subsets with $0 < \text{diam } A \leq \text{diam } B < \infty$, then $\text{diam } f(B)$ is finite and*

$$\frac{1}{2\eta(\text{diam } B/\text{diam } A)} \leq \frac{\text{diam } f(A)}{\text{diam } f(B)} \leq \eta \left(\frac{2 \text{diam } A}{\text{diam } B} \right).$$

- (iii) *The map f sends Cauchy sequences to Cauchy sequences, and there is a unique extension of f to an η -quasisymmetric homeomorphism $\bar{f} : \bar{X} \rightarrow \bar{Y}$.*

A homeomorphism of metric spaces $f : X \rightarrow Y$ is called quasi-Möbius if there is a homeomorphism $\theta : [0, \infty) \rightarrow [0, \infty)$ such that for all quadruples $x_1, x_2, x_3, x_4 \in X$ of distinct points, the following relationship holds:

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \theta([x_1, x_2, x_3, x_4]),$$

where the cross ratio is denoted by

$$[x, y, z, w] := \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

We will use the same notational conventions for quasi-Möbius maps as for quasisymmetric maps. The inverse of a quasi-Möbius homeomorphism is again quasi-Möbius, and a Möbius transformation of \mathbb{R}^n is θ -quasi-Möbius with $\theta(t) = t$. Quasi-Möbius maps need not send bounded sets to bounded sets. The following result of Väisälä [20, Theorems 3.2 and 3.10] shows that this is the only essential difference between quasi-Möbius and quasisymmetric homeomorphisms.

THEOREM 2.2 (Väisälä). *Let $f : X \rightarrow Y$ be a homeomorphism of metric spaces. If f is η -quasisymmetric, then it is θ -quasi-Möbius with θ depending only on η . If X is unbounded and f is a θ -quasi-Möbius homeomorphism which maps unbounded sequences to unbounded sequences, then f is θ -quasisymmetric.*

Let $\lambda > 1$. A metric space (X, d) is λ -linearly locally connected (λ -LLC) if for all $a \in X$ and $r > 0$ the following conditions are satisfied.

(λ -LLC₁) For each pair of distinct points $x, y \in B(a, r)$, there is a continuum $E \subseteq B(a, \lambda r)$ such that $x, y \in E$.

(λ -LLC₂) For each pair of distinct points $x, y \in X - B(a, r)$, there is a continuum $E \subseteq X - B(a, r/\lambda)$ such that $x, y \in E$.

Recall that a continuum is a connected, compact set containing more than one point. Note that we do not place any upper restriction on the radius r in this definition, though the λ -LLC₂ condition is vacuously true for $r > \text{diam}(X, d)$.

REMARK 2.3. The terminology ‘linearly locally connected’ is justified by the following observation. Suppose that (X, d) is a λ -LLC metric space, $x \in X$, and $r > 0$. Let $C(x)$ be the connected component of $B(x, r)$ containing x . Then $B(x, r/\lambda) \subseteq C(x) \subseteq B(x, r)$.

Väisälä proved that the LLC condition is preserved by quasi-Möbius homeomorphisms [20, Theorems 4.4 and 4.5]. In light of Theorem 2.2, we may state the following theorem.

THEOREM 2.4 (Väisälä). *If X is a λ -LLC metric space and If $f : X \rightarrow Y$ is an η -quasisymmetric or η -quasi-Möbius homeomorphism, then Y is λ' -LLC for some λ' depending only on λ and η .*

REMARK 2.5. Each of the spaces \mathbb{S}^2 , \mathbb{S}^{2*} , \mathbb{D}^2 , \mathbb{R}^2 , and \mathbb{R}_+^2 is LLC. This, along with Theorem 2.4, proves the final statement of Theorem 1.2.

The question of which planar domains are quasisymmetrically equivalent to \mathbb{D}^2 was essentially answered Ahlfors in [1]. The result was stated in terms of the LLC condition by Gehring in [9].

THEOREM 2.6 (Ahlfors, Gehring). *Let $D \subseteq \mathbb{S}^2$ be a domain which is LLC when endowed with the standard metric, and such that ∂D is connected and contains at least two points. Then there exists a quasimetric homeomorphism $f : D \rightarrow \mathbb{D}^2$ with distortion function depending only on the LLC constant of D .*

Let I be any connected subset of \mathbb{R} . For any subset $U \subseteq \bar{X}$, we call a continuous map $\gamma : I \rightarrow U$ a path in U . If the path γ happens to be an embedding, then we call the image of γ an arc in U . We will make repeated use of the fact that the image of any path is arc-connected. A path γ is called proper if for any compact set $K \subseteq U$, the pre-image $\gamma^{-1}(K)$ is compact. The image of a path γ will be denoted by $\text{im } \gamma$.

If X is locally path-connected, we will often employ a condition similar to LLC which uses arcs instead of continua. This condition extends to the completion \bar{X} in a particularly nice way. We say that a locally compact metric space (X, d) is λ -LLC if for all $a \in \bar{X}$ and $r > 0$ the following conditions are satisfied.

(λ -LLC₁) For each pair of distinct points $x, y \in B_{\bar{X}}(a, r)$, there is an embedding $\gamma : [0, 1] \rightarrow \bar{X}$ such that $\gamma(0) = x$, $\gamma(1) = y$, $\gamma|_{(0,1)} \subseteq X$, and $\gamma \subseteq B_{\bar{X}}(a, \lambda r)$.

(λ -LLC₂) For each pair of distinct points $x, y \in \bar{X} - B_{\bar{X}}(a, r)$, there is an embedding $\gamma : [0, 1] \rightarrow \bar{X}$ such that $\gamma(0) = x$, $\gamma(1) = y$, $\gamma|_{(0,1)} \subseteq X$, and $\gamma \subseteq \bar{X} - B_{\bar{X}}(a, r/\lambda)$.

If a metric space X is λ -LLC, then it is also λ -LLC. The next proposition states that the two conditions are quantitatively equivalent for the spaces in consideration in this paper.

PROPOSITION 2.7. *Let (X, d) be a locally compact, locally path-connected, and λ -LLC metric space. Then X is λ' -LLC, where λ' depends only on λ . In particular, the space \bar{X} is λ' -LLC.*

Proof. The key ingredient is the following statement: If $U \subseteq X$ is an open subset of X , and $E \subseteq U$ is a continuum, then any pair of points $x, y \in E$ are contained in an arc in U . The details are straightforward and left to the reader. \square

The LLC condition allows a useful addition to Remark 2.3.

LEMMA 2.8. *Let (X, d) be a λ -LLC metric space, $p \in \partial X$, and $\epsilon > 0$. Then there is a connected subset $C \subseteq X$ which is closed in X , such that*

$$B_{\bar{X}}(p, \epsilon/\lambda) \cap X \subseteq C \subseteq \bar{B}_{\bar{X}}(p, \epsilon) \cap X.$$

Proof. Define

$$S = \{(x, y) \in X \times X : x \neq y \text{ and } x, y \in B_{\bar{X}}(p, \epsilon/\lambda)\} \quad \text{and} \quad C_0 = \bigcup_{(x,y) \in S} \gamma_{x,y},$$

where $\gamma_{x,y}$ is the arc connecting x to y provided by the λ -LLC condition. Taking C to be the closure of C_0 in X proves the lemma. \square

Let (X, d) be a metric space. For any $Q \geq 0$, we define the Q -dimensional Hausdorff measure of a subset $E \subseteq X$ by

$$\mathcal{H}_d^Q(E) := \lim_{\epsilon \rightarrow 0} \mathcal{H}_d^{Q,\epsilon}(E),$$

where $\mathcal{H}_d^{Q,\epsilon}(E)$ is the Carathéodory pre-measure defined as follows. Let \mathcal{B}_ϵ be the collection of all covers \mathcal{C} of E by closed balls of radius less than ϵ . Then

$$\mathcal{H}_d^{Q,\epsilon}(E) := \inf_{\mathcal{C} \in \mathcal{B}_\epsilon} \sum_{B \in \mathcal{C}} (\text{radius}(B))^Q.$$

When computing $\mathcal{H}_d^{Q,\epsilon}(E)$ it suffices to consider covers of E by balls centered in the ϵ -neighborhood of E . For a full description of Hausdorff measure and the Carathéodory construction, see [8, Chapter 2.10]. Note that our definition differs from that in literature as we sum radii of balls rather than diameters of arbitrary closed sets; the resulting measures are comparable and thus equivalent for our purposes.

A metric space (X, d) is called Ahlfors Q -regular, $Q \geq 0$, if there exists a constant $K \geq 1$ such that for all $a \in X$ and $0 < r \leq \text{diam } X$, we have

$$\frac{r^Q}{K} \leq \mathcal{H}^Q(\bar{B}_d(a, r)) \leq Kr^Q. \quad (2.1)$$

REMARK 2.9. This is the definition used by Semmes in [16], except that we do not require X to be complete. In [4], Bonk and Kleiner use the slightly weaker condition that for all $a \in X$ and $0 < r \leq \text{diam } X$,

$$\frac{r^Q}{K} \leq \mathcal{H}^Q(B_d(a, r)) \leq Kr^Q. \quad (2.2)$$

The condition (2.1) implies the condition (2.2) with the same constant. In the case that X is unbounded, the two conditions are equivalent. The main reason to use (2.1) is that it implies that

$$\mathcal{H}^Q(X) \leq K(\text{diam } X)^Q,$$

and so even for $r > \text{diam } X$ we have the upper bound

$$\mathcal{H}^Q(\bar{B}_d(a, r)) \leq Kr^Q.$$

This is not necessarily true for spaces which only satisfy the weaker condition (2.2).

A metric space (X, d) is called M -doubling if for every $a \in X$ and all $r > 0$, the open ball $B(a, r)$ can be covered by at most M balls of radius $r/2$. The next proposition lists some useful properties of Ahlfors Q -regular spaces.

PROPOSITION 2.10. *Let $Q \geq 0$, and let (X, d) be an Ahlfors Q -regular metric space with constant K . Then the following statements hold.*

- (i) *The metric space X is M -doubling where M depends only on K and Q .*
- (ii) *Any bounded subset of X is totally bounded.*
- (iii) *The completion \bar{X} is Ahlfors Q -regular with constant K' depending only on K and Q .*

In the proof of Proposition 2.10, we will need the following basic covering lemma, which is proven and discussed in [11, Chapter 1].

LEMMA 2.11. *Let (X, d) be a metric space. Suppose that $\{B(x_i, r_i)\}_{i \in I}$ is a collection of balls in X of uniformly bounded radius. Then there exists a subset $J \subseteq I$ such that*

$$\bigcup_{i \in I} B(x_i, r_i) \subseteq \bigcup_{i \in J} B(x_i, 5r_i),$$

and

$$B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$$

for distinct indices i and j in J .

Proof of Proposition 2.10. The proof of statement (i) is well known and can be found, in particular, in [16, Chapter 2.2]. Statement (ii) follows directly from statement (i). We will prove (iii). Let $a \in \bar{X}$ and let $r \leq \text{diam } \bar{X} = \text{diam } X$. We first consider the case that $a \in X$. Let $\epsilon > 0$ and consider any cover $\mathcal{B} = \{\bar{B}_{\bar{X}}(x_i, r_i)\}_{i \in I}$ of $\bar{B}_{\bar{X}}(a, r)$ such that $r_i < \epsilon$. If the center x_i of a covering ball happens to be in the boundary ∂X , then let x'_i be any point in $\bar{B}_{\bar{X}}(x_i, r_i) \cap X$. For those x_i which are not boundary points, let $x'_i = x_i$. Then for all $i \in I$, we have

$$\bar{B}_{\bar{X}}(x_i, r_i) \subseteq \bar{B}_{\bar{X}}(x'_i, 2r_i).$$

As a result, the collection

$$\mathcal{B}' = \{\bar{B}_{\bar{X}}(x'_i, 2r_i)\}_{i \in I}$$

is a cover of $\bar{B}_{\bar{X}}(a, r)$ by balls centered in X of radius less than 2ϵ . To prove the lower bound, we note that this implies that the collection $\{\bar{B}_X(x'_i, 2r_i)\}_{i \in I}$ is a cover of $\bar{B}_X(a, r)$ by balls in X of radius less than 2ϵ . As a result,

$$\mathcal{H}_X^{Q, 2\epsilon}(\bar{B}_X(a, r)) \leq \sum_{i \in I} (2r_i)^Q \leq 2^Q \sum_{i \in I} r_i^Q.$$

Since the cover \mathcal{B} was arbitrary for the purposes of calculating $\mathcal{H}_{\bar{X}}^{Q, \epsilon}(\bar{B}_{\bar{X}}(a, r))$, letting ϵ tend to zero yields

$$\mathcal{H}_X^Q(\bar{B}_X(a, r)) \leq 2^Q \mathcal{H}_{\bar{X}}^Q(\bar{B}_{\bar{X}}(a, r)).$$

The Q -regularity of X now implies that

$$\frac{r^Q}{2^Q K} \leq \mathcal{H}_{\bar{X}}^Q(\bar{B}_{\bar{X}}(a, r)). \quad (2.3)$$

To show the upper bound, we apply Lemma 2.11 to the collection \mathcal{B}' . Let $\{\bar{B}_{\bar{X}}(x'_i, 10r_i)\}_{i \in J}$ be the resulting cover of $\bar{B}_{\bar{X}}(a, r)$. Now

$$\mathcal{H}_{\bar{X}}^{Q, 10\epsilon}(\bar{B}_{\bar{X}}(a, r)) \leq \sum_{i \in J} (10r_i)^Q \leq 5^Q \sum_{i \in J} (2r_i)^Q. \quad (2.4)$$

For sufficiently small values of ϵ , we have $\bar{B}_X(x'_i, 2r_i) \subseteq \bar{B}_X(a, 2r)$ for each $i \in J$. Thus by (2.4), the Ahlfors Q -regularity of X , and the disjointness provided by the covering lemma, we have

$$\mathcal{H}_{\bar{X}}^{Q, 10\epsilon}(\bar{B}_{\bar{X}}(a, r)) \leq 5^Q \sum_{i \in J} K \mathcal{H}_X^Q(\bar{B}_X(x'_i, 2r_i)) \leq 5^Q K \mathcal{H}_X^Q(\bar{B}_X(a, 2r)).$$

Letting ϵ tend to zero and applying the Ahlfors Q -regularity of X yields

$$\mathcal{H}_{\bar{X}}^Q(\bar{B}_{\bar{X}}(a, r)) \leq 10^Q K^2 r^Q. \quad (2.5)$$

Note that (2.5) holds if $r > \text{diam } X$ as well.

If $a \in \partial X$, then we may find $a' \in X$ such that $d(a, a') < r/2$. Then

$$\bar{B}_{\bar{X}}\left(a', \frac{r}{2}\right) \subseteq \bar{B}_{\bar{X}}(a, r) \subseteq \bar{B}_{\bar{X}}(a', 2r).$$

Then using (2.3) and (2.5), we have

$$\frac{r^Q}{4^Q K} \leq \mathcal{H}_X^Q(\bar{B}_X(a', r/2)) \leq \mathcal{H}_X^Q(\bar{B}_X(a, r)) \leq \mathcal{H}_X^Q(\bar{B}_X(a', 2r)) \leq 20^Q K^2 r^Q. \quad (2.6)$$

The estimates (2.3), (2.5), and (2.6) show that \bar{X} is Ahlfors Q -regular with constant $20^Q K^2$.

Given a λ -LLC metric space (X, d) which is Ahlfors Q -regular with constant K , we define the data of (X, d) to be the triple (λ, K, Q) . Propositions 2.10 and 2.7 show that if (X, d) is an Ahlfors Q -regular and LLC metric space, then \bar{X} is also Ahlfors Q -regular and LLC with data depending only on the data of X .

3. The sphere and punctured sphere

Proof of Theorem 1.2(i). This is merely a restatement of Theorem 1.1. Suppose that X is a complete, bounded, Ahlfors 2-regular, and LLC metric space homeomorphic to the sphere or the plane. By Proposition 2.10(ii), X is totally bounded, and thus compact. Accordingly, X is homeomorphic to \mathbb{S}^2 and satisfies the hypotheses of Theorem 1.1, which provides an η -quasisymmetric homeomorphism $f : X \rightarrow \mathbb{S}^2$, where η depends only on the data of X .

Proof of Theorem 1.2(ii). Suppose that X is a bounded, Ahlfors 2-regular, LLC metric space such that $\text{card } \partial X = 1$. Since X is not complete, it cannot be compact, and so it must be homeomorphic to \mathbb{R}^2 . By Proposition 2.10(ii) and (iii), and Proposition 2.7, \bar{X} is a compact, Ahlfors 2-regular, and LLC metric space with data depending only on the data of X . A standard theorem of point-set topology [13, Theorem 29.1] implies that \bar{X} is homeomorphic to the one-point compactification of X , namely \mathbb{S}^2 . Theorem 1.1 now implies that there is an η -quasisymmetric homeomorphism $f : \bar{X} \rightarrow \mathbb{S}^2$, where η depends only on the data of X . The restriction $f|_X$ is an η -quasisymmetric homeomorphism from X to \mathbb{S}^{2*} .

4. The boundary of a disk

By Proposition 2.1(ii), a necessary condition for a metric space X to be quasisymmetrically equivalent to \mathbb{D}^2 is that ∂X is a quasicircle, that is, the quasisymmetric image of the circle \mathbb{S}^1 . In this section, we prove Theorem 1.3, which provides sufficient conditions on X for the boundary ∂X to be a quasicircle.

REMARK 4.1. Let X be as in the assumptions of Theorem 1.2(iii); that is, X is homeomorphic to the plane, Ahlfors 2-regular, LLC, bounded, and satisfies $\text{card } \partial X \geq 2$. By Proposition 2.10, the completion \bar{X} is doubling. As a result, \bar{X} is compact and ∂X is doubling. Thus Theorem 1.3 allows us to conclude that ∂X is a quasicircle. For the proof of Theorem 1.2(iii), we will need only the weaker conclusion that ∂X is homeomorphic to the circle.

As mentioned in the introduction, we will show that ∂X is homeomorphic to the circle by showing it is a locally connected metric continuum such that the removal of any one point does not disconnect the space, while the removal of any two does. We begin by giving the purely topological results which will be used in the proof.

DEFINITION 4.2. Let X be a topological space, and let $k \in \mathbb{N}$. A subset $U \subseteq X$ is k -ended if for every compact subset $K \subseteq X$, there is another compact subset $K' \supseteq K$ such that $U - K'$

has exactly k components. If, in addition, each of these components are arc-connected, then we say that U is *arc- k -ended*.

REMARK 4.3. A trivial but useful example is the following: if X is a topological space homeomorphic to the disk and $C \subseteq X$ is a compact subset, then $X - C$ is arc-1-ended.

LEMMA 4.4. *Let X be a topological space homeomorphic to the disk, and suppose that $\gamma : (0, 1) \rightarrow X$ is a proper embedding. Then $X - \text{im } \gamma$ has exactly two components, each of which is arc-connected and arc-1-ended. Furthermore, there exists an ascending sequence $K_1 \subseteq K_2 \subseteq \dots$ of compact subsets of X with $\bigcup_{n \in \mathbb{N}} K_n = X$ such that for each component U of $X - \text{im } \gamma$ and each $n \in \mathbb{N}$, $U - K_n$ is arc-connected.*

Proof. As this is a purely topological result, we may assume that X is \mathbb{S}^{2*} , with the puncture at a point labeled ∞ . The assumption that γ is proper now means that $\text{im } \gamma \cup \{\infty\}$ defines a Jordan curve in \mathbb{S}^2 . By the Schönflies theorem, there is a homeomorphism $\Theta : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ mapping $\text{im } \gamma \cup \{\infty\}$ to a great circle C . Thus, up to homeomorphism, $X - \text{im } \gamma$ is the complement of a line in \mathbb{R}^2 . In this case, the assertions of the lemma are clear. \square

LEMMA 4.5. *Let X be a topological space homeomorphic to the disk, and let γ and γ' be proper embeddings of $(0, 1)$ into X . Suppose that there is a compact interval $I \subseteq (0, 1)$ such that $\gamma(t) = \gamma'(t)$ for all $t \in (0, 1) - I$. Then there is a compact subset K of X such that if $p, q \in X - K$ are in different components of $X - \text{im } \gamma'$, then they are in different components of $X - \text{im } \gamma$.*

Proof. Let U be a component of $X - \text{im } \gamma$. By Lemma 4.4, we may find a compact set K such that $\gamma(I) \subseteq K$ and $U - K$ is arc-connected. It suffices to show that if p and q are points in $U - K$, then they may be connected in $X - \text{im } \gamma'$. By assumption, p and q may be joined by an arc α which meets neither K nor $\text{im } \gamma$. This implies that α does not meet $\text{im } \gamma'$ either, and so p and q are in the same component of $X - \text{im } \gamma'$. \square

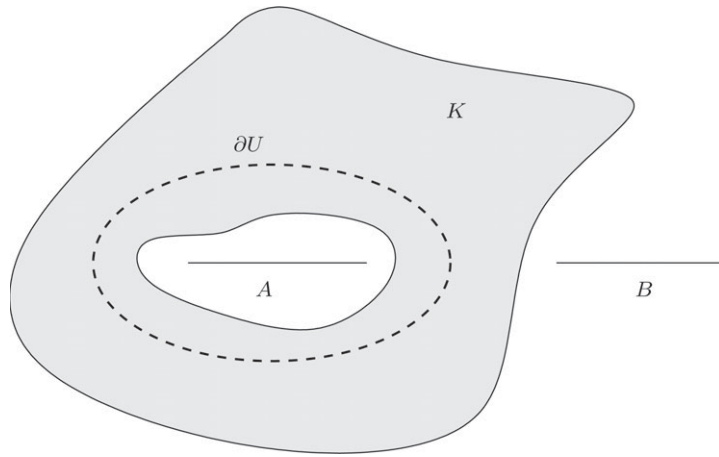
A proof of the following separation theorem may be found in [14, Chapter V.2].

LEMMA 4.6 (Janiszewski's theorem). *Let X be a topological space homeomorphic to the disk. Suppose that $K \subseteq X$ is compact and connected, $C \subseteq X$ is closed and connected, and $C \cap K$ is connected. If x and y are points in X which lie in the same component of $X - C$ and in the same component of $X - K$, then they also lie in the same component of $X - (C \cup K)$.*

We now turn to the proof of Theorem 1.3. The LLC condition is not needed to show that the boundary is a continuum.

PROPOSITION 4.7. *If X is a metric space homeomorphic to the disk such that \bar{X} is compact, then ∂X is a continuum if it contains at least two points.*

Proof. As a closed subset of the compact space \bar{X} , the boundary ∂X is compact. Assuming that ∂X contains at least two points, it suffices to show that ∂X is connected. If ∂X is

FIGURE 1. The situation if ∂X is not connected.

not connected, we may find disjoint, non-empty, and closed subsets A and B of ∂X with $A \cup B = \partial X$. There is some $\epsilon > 0$ such that $\text{dist}(A, B) > 2\epsilon$. Let U and V be the open ϵ -neighborhoods of A and B respectively; then $U \cap V$ is empty. Each point of A is in the interior of U by definition, and each point of B is at a distance at least ϵ from U . Thus $\partial U \cap \partial X = \emptyset$, and so ∂U is a compact subset of X . By Remark 4.3, there is a compact set $K \subseteq X$ containing ∂U such that each pair of points $u, v \in X - K$ can be connected by an arc that does not intersect ∂U (Figure 1).

Let $\delta = \text{dist}(K, \partial X)$, and let $\epsilon' < \min(\delta/2, \epsilon/2)$. We may find points u and v of X in $\mathcal{N}_{\epsilon'}(A)$ and $\mathcal{N}_{\epsilon'}(B)$, respectively. Then $u \in U$ and $v \in V$ but neither is in K . Thus they may be connected by an arc which does not intersect ∂U , contradicting the fact that U and V are disjoint. \square

Throughout the rest of this section, we will assume that X is a λ -LLC metric space homeomorphic to the disk such that \bar{X} is compact and ∂X contains at least two points. Proposition 2.7 shows that we have lost no generality in doing so.

PROPOSITION 4.8. *For each $p \in \partial X$, $\partial X - \{p\}$ is connected.*

Proof. We argue by way of contradiction. Suppose that A and B are disjoint, non-empty, and relatively closed subsets of $\partial X - \{p\}$ satisfying $A \cup B = \partial X - \{p\}$. We may find disjoint open sets $U, V \subseteq \bar{X}$ containing A and B , respectively. Choose $\epsilon_1 > 0$ so small that we may find points $a \in A - B_{\bar{X}}(p, \epsilon_1)$ and $b \in B - B_{\bar{X}}(p, \epsilon_1)$. Let

$$\epsilon_2 = \frac{\epsilon_1}{2\lambda(4\lambda + 1)}.$$

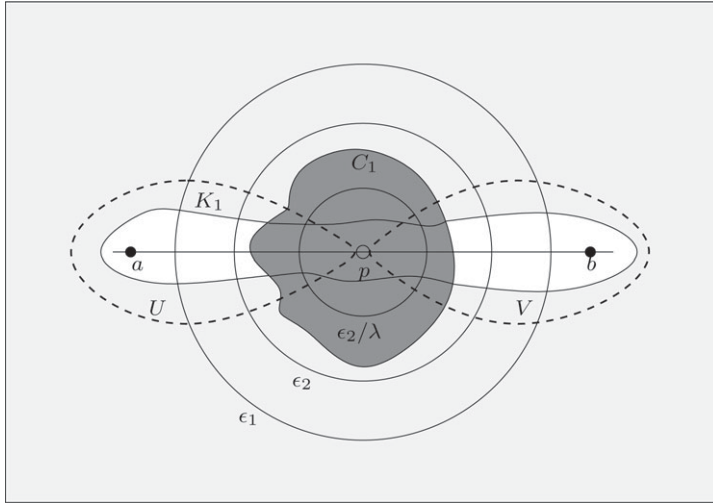
Then $X - (U \cup V \cup B_{\bar{X}}(p, \epsilon_2/\lambda))$ is a compact subset of X . Because X is homeomorphic to the plane, there is a topological closed disk $K_1 \subseteq X$ such that

$$K_1 \supseteq X - (U \cup V \cup B_{\bar{X}}(p, \epsilon_2/\lambda)).$$

Note that K_1 is a compact and connected subset of X .

By Lemma 2.8, there is a closed and connected subset $C_1 \subseteq X$ such that

$$B_{\bar{X}}(p, \epsilon_2/\lambda) \cap X \subseteq C_1 \subseteq \bar{B}_{\bar{X}}(p, \epsilon_2) \cap X.$$

FIGURE 2. The situation if $C_1 \cap K_1$ is not connected.

We would like to apply Lemma 4.6 to C_1 and K_1 , but it may be the case that $C_1 \cap K_1$ is not connected (Figure 2). If $C_1 \cap K_1$ is connected, then set $C = C_1$ and $K = K_1$. Otherwise, we will add ‘connectors’ to each component of $C_1 \cap K_1$. Set $\delta = \text{dist}(C_1 \cap K_1, \partial X)$; note that $\delta \leq \epsilon_2$. By compactness there is a cover of $C_1 \cap K_1$ by a finite collection of balls $\{B_i\}$, where $B_i := B_X(x_i, \delta/2\lambda)$ with $x_i \in C_1 \cap K_1$. Let $C(x_i)$ denote the closure in X of the component of x_i in $B_X(x_i, \delta/2)$. By Remark 2.3, $B_i \subseteq C(x_i)$, and so the collection $\{C(x_i)\}$ is a cover of $C_1 \cap K_1$ by finitely many compact and connected sets in X . For each pair of distinct indices i, j , the λ -LLC property provides an arc γ_{ij} in X connecting x_i to x_j inside $B_X(x_i, 2\lambda d(x_i, x_j))$. Since $C_1 \subseteq \bar{B}_X(p, \epsilon_2)$, we have

$$\gamma_{ij} \subseteq B_X(x_i, 2\lambda d(x_i, x_j)) \subseteq B_X(x_i, 4\lambda \epsilon_2) \subseteq B_X(p, \epsilon_1/2\lambda) \cap X.$$

We also have

$$\bigcup_i C(x_i) \subseteq \bigcup_i B_X(x_i, \epsilon_2/2) \subseteq B_X(p, \epsilon_1/2\lambda) \cap X.$$

Let

$$K = K_1 \cup \bigcup_i C(x_i) \cup \bigcup_{i \neq j} \gamma_{ij} \quad \text{and} \quad C = C_1 \cup \bigcup_i C(x_i) \cup \bigcup_{i \neq j} \gamma_{ij}.$$

Now we have that K is compact and connected, C is closed and connected, and $C \cap K$ is connected. As K is compact and the points a and b are in the boundary ∂X , Remark 4.3 implies that we may find points u and v in the same component of $X - K$ such that

$$u \in U \cap (X - B_X(p, \epsilon_1/2)) \quad \text{and} \quad v \in V \cap (X - B_X(p, \epsilon_1/2)).$$

Furthermore, $C \subseteq B_X(p, \epsilon_1/2\lambda) \cap X$, and so by λ -LLC₂ we see that u and v are in the same component of $X - C$. Therefore, Lemma 4.6 implies that u and v are in the same component of $X - (C \cup K)$. However,

$$C \cup K \supseteq (B_X(p, \epsilon_2/\lambda) \cap X) \cup (X - (U \cup V \cup B_X(p, \epsilon_2/\lambda))) \supseteq X - (U \cup V).$$

This means that u and v lie in a connected subset of $U \cup V$, which contradicts the facts that $u \in U$, $v \in V$, and $U \cap V = \emptyset$. Thus $\partial X - \{p\}$ is connected. \square

DEFINITION 4.9. Let p and q be distinct points in ∂X . A *crosscut* connecting p and q is an embedding $\gamma : [0, 1] \rightarrow X \cup \{p, q\}$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Note that if γ is a crosscut, then $\gamma : (0, 1) \rightarrow X$ is a proper embedding.

LEMMA 4.10. Let γ be any crosscut, and let U and V be the components of $X - \text{im } \gamma$. The following statements hold:

- (i) $\bar{X} = \bar{U} \cup \bar{V}$;
- (ii) $\bar{U} - \text{im } \gamma$ and $\bar{V} - \text{im } \gamma$ are the components of $\bar{X} - \text{im } \gamma$;
- (iii) $\bar{U} \cap \partial X$ and $\bar{V} \cap \partial X$ are connected.

Proof. (i) This follows immediately from the definitions.

(ii) To show that $\bar{U} - \text{im } \gamma$ and $\bar{V} - \text{im } \gamma$ are the components of $\bar{X} - \text{im } \gamma$, we will show that they are each relatively closed in $\bar{X} - \text{im } \gamma$, connected, they do not intersect, and their union is all of $\bar{X} - \text{im } \gamma$. Clearly they are relatively closed, and by (i) we have their union is all of $\bar{X} - \text{im } \gamma$. By definition, U is a connected subset of $\bar{X} - \text{im } \gamma$. Since $\bar{U} - \text{im } \gamma$ is the closure of U in $\bar{X} - \text{im } \gamma$, it follows that $\bar{U} - \text{im } \gamma$ is connected. The same argument shows that $\bar{V} - \text{im } \gamma$ is connected.

Suppose that $z \in \bar{U} \cap \bar{V} - \text{im } \gamma$. Then there is an $\epsilon > 0$ such that $B_{\bar{X}}(z, \epsilon) \cap \text{im } \gamma = \emptyset$. We may find points $u \in U$ and $v \in V$ such that $u, v \in B_{\bar{X}}(z, \epsilon/\lambda)$. By the λ -LLC condition, we may connect u to v inside $B_{\bar{X}}(z, \epsilon)$, and hence without intersecting $\text{im } \gamma$. This contradicts the assumption that u and v are in different components of $X - \text{im } \gamma$. Thus $\bar{U} - \text{im } \gamma$ and $\bar{V} - \text{im } \gamma$ do not intersect.

(iii) We now show that $\bar{U} \cap \partial X$ is connected; the same argument will apply to $\bar{V} \cap \partial X$. Suppose that C and D are disjoint, non-empty, and closed subsets of $\bar{U} \cap \partial X$ such that $C \cup D = \bar{U} \cap \partial X$. Then we may find an $\epsilon > 0$ such that $\text{dist}(C, D) > \epsilon$.

We claim that there is some $\delta > 0$ such that

$$U \cap \mathcal{N}_\delta(\partial X) \subseteq \mathcal{N}_\epsilon(C) \cup \mathcal{N}_\epsilon(D).$$

If not, then for all $n \in \mathbb{N}$, we may find points $u_n \in U$ and $x_n \in \partial X$ such that

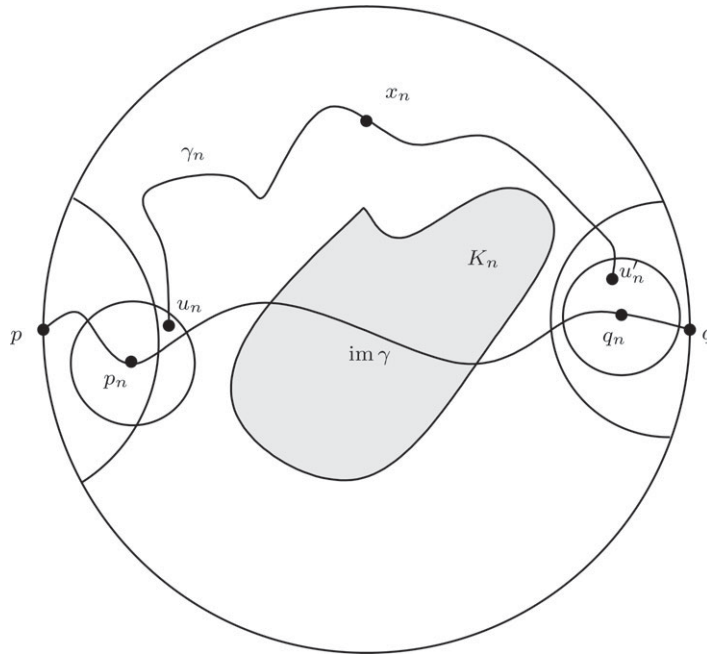
$$d(u_n, x_n) < \frac{1}{n} \quad \text{and} \quad \text{dist}(x_n, C \cup D) \geq \text{dist}(u_n, C \cup D) - d(u_n, x_n) \geq \epsilon - \frac{1}{n}.$$

As ∂X is compact, there is a subsequence $\{x_{n_k}\}$ converging to a point $x \in \partial X - (\mathcal{N}_\epsilon(C) \cup \mathcal{N}_\epsilon(D))$. By continuity, we find that $d(x, C \cup D) \geq \epsilon$, and so $x \notin (\mathcal{N}_\epsilon(C) \cup \mathcal{N}_\epsilon(D))$. However, $\{u_{n_k}\}$ also converges to x , contradicting the assumption that $\bar{U} \cap \partial X = C \cup D$, and the claim is proven.

Consider that $K := X - \mathcal{N}_\delta(\partial X)$ is a compact subset of X . By Lemma 4.4, U is one-ended, and so we may find a compact subset $K' \supseteq K$ such that $U - K'$ is connected. However, the claim shows that $\mathcal{N}_\epsilon(C)$ and $\mathcal{N}_\epsilon(D)$ constitute a cover of $U - K'$ by non-empty, disjoint, open sets. This is a contradiction, and so $\bar{U} \cap \partial X$ must be connected. \square

PROPOSITION 4.11. For any pair of distinct points $p, q \in \partial X$, the set $\partial X - \{p, q\}$ is not connected.

Proof. The λ -LLC condition provides a crosscut γ connecting p to q . Let U and V be the components of $X - \text{im } \gamma$, and let $A = \bar{U} \cap \partial X - \{p, q\}$ and $B = \bar{V} \cap \partial X - \{p, q\}$. Then A and B are relatively closed subsets of $\partial X - \{p, q\}$. It follows from 4.10(i) that $A \cup B = \partial X - \{p, q\}$. We will show $A \cap B = \emptyset$ and that A and B are non-empty.

FIGURE 3. The proof that A is non-empty.

Suppose that there is some point $z \in A \cap B$. Then $z \notin \text{im } \gamma$, and so there is some $\epsilon > 0$ such that $B_{\bar{X}}(z, \epsilon) \cap \text{im } \gamma = \emptyset$. By assumption, we may find points $u \in U$ and $v \in V$ contained in $B_{\bar{X}}(z, \epsilon/\lambda)$. By the λ -LLC condition, there is an arc connecting u to v which lies in $B_{\bar{X}}(z, \epsilon/\lambda)$, and hence does not intersect $\text{im } \gamma$. This is a contradiction, and so $A \cap B = \emptyset$.

By symmetry, it suffices to show that A is non-empty (Figure 3). Lemma 4.4 provides an exhaustion $\{K_n\}_{n \in \mathbb{N}}$ of X by compact sets such that for each $n \in \mathbb{N}$, $U - K_n$ is arc-connected. For each $n \in \mathbb{N}$, we may find points

$$p_n \in \text{im } \gamma \cap (X - K_n) \cap B_{\bar{X}}(p, 1/n) \quad \text{and} \quad q_n \in \text{im } \gamma \cap (X - K_n) \cap B_{\bar{X}}(q, 1/n).$$

Because each K_n is compact, there is a sequence of positive numbers $\{\epsilon_n\}_{n \in \mathbb{N}}$ tending to zero such that $B_X(p_n, \epsilon_n) \cap K_n = \emptyset$ and $B_X(q_n, \epsilon_n) \cap K_n = \emptyset$. Each point on $\text{im } \gamma$ is a limit point of U , so there are points $u_n \in B_X(p_n, \epsilon_n) \cap U$ and $u'_n \in B_X(q_n, \epsilon_n) \cap U$. We may connect u_n to u'_n via an arc γ_n in $U - K_n$. By the connectedness of γ_n , there is a point $x_n \in \gamma_n \cap U$ such that

$$d(u_n, x_n) = \frac{d(u_n, u'_n)}{2} \leq d(u'_n, x_n).$$

By the compactness of \bar{X} , $\{x_n\}_{n \in \mathbb{N}}$ subconverges to some point $x \in \bar{X}$. Since $\bigcup_{n \in \mathbb{N}} K_n = X$, we have $x \in \partial X$. Furthermore, $u_n \rightarrow p$ and $u'_n \rightarrow q$ as $n \rightarrow \infty$, so

$$d(p, x) = \frac{d(p, q)}{2} \leq d(q, x).$$

Thus $x \in \partial X - \{p, q\}$. Since $\{x_n\} \subseteq U$, we have $x \in A$. \square

LEMMA 4.12. *Let a, b, p, q be distinct points in ∂X , and let γ and γ' be crosscuts connecting a and b . Suppose that there is a compact interval $I \subseteq (0, 1)$ such that $\gamma(t) = \gamma'(t)$ for all*

$t \in [0, 1] - I$. If p and q are in different components of $\bar{X} - \text{im } \gamma'$, then they are in different components of $\bar{X} - \text{im } \gamma$.

Proof. By Lemma 4.5, there is a compact set K such that if p' and q' are points of $X - K$ which are in different components of $X - \text{im } \gamma'$, then they are in different components of $X - \text{im } \gamma$. We may find $\epsilon > 0$ such that

$$B_{\bar{X}}(p, \epsilon) \cap (\text{im } \gamma' \cup \text{im } \gamma \cup K) = \emptyset$$

Let $p' \in X$ such that $d(p, p') < \epsilon/\lambda$. Then $p' \notin K$. The λ -LLC condition provides an arc α connecting p to p' which does not intersect $\text{im } \gamma \cup \text{im } \gamma'$. Thus p and p' are in the same component of $\bar{X} - \text{im } \gamma'$ and the same component of $\bar{X} - \text{im } \gamma$. Similarly, we may find $q' \in X - K$ such that q and q' are in the same component of $\bar{X} - \text{im } \gamma'$ and the same component of $\bar{X} - \text{im } \gamma$. Thus, since p and q are in different components of $\bar{X} - \gamma'$, we see that p' and q' are in different components of $\bar{X} - \text{im } \gamma'$. This implies that p' and q' are in different components of $X - \text{im } \gamma$. Lemma 4.10(ii) allows us to conclude that p' and q' are in different components of $\bar{X} - \text{im } \gamma$, and hence so are p and q . \square

LEMMA 4.13. *Let a, b, p, q be distinct points in ∂X . Let γ_{pq} be a crosscut connecting p to q , and let γ_{ab} be a crosscut connecting a to b . If a and b are in different components of $\bar{X} - \text{im } \gamma_{pq}$, then p and q are in different components of $\bar{X} - \text{im } \gamma_{ab}$.*

Proof. The fact that the points a, b, p, q are all distinct implies that $K = \gamma_{a,b} \cap \gamma_{p,q}$ is a compact subset of X . We may assume that K is non-empty. For ease of notation, we identify \mathbb{R}^2 with \mathbb{C} in this proof. Let $\phi : X \rightarrow \mathbb{C}$ be any homeomorphism. We may find $R > 0$ such that $\phi(K) \subseteq B_{\mathbb{C}}(0, R)$. As $\gamma_{ab}|_{(0,1)}$ and $\gamma_{pq}|_{(0,1)}$ are proper, we may find parameters $t_1, t_2, s_1, s_2 \in (0, 1)$ such that

$$\begin{aligned} t_1 &= \min\{t \in (0, 1) : |\phi \circ \gamma_{ab}(t)| = R\}, \\ t_2 &= \max\{t \in (0, 1) : |\phi \circ \gamma_{ab}(t)| = R\}, \\ s_1 &= \min\{s \in (0, 1) : |\phi \circ \gamma_{pq}(s)| = R\}, \\ s_2 &= \max\{s \in (0, 1) : |\phi \circ \gamma_{pq}(s)| = R\}. \end{aligned}$$

Consider the points

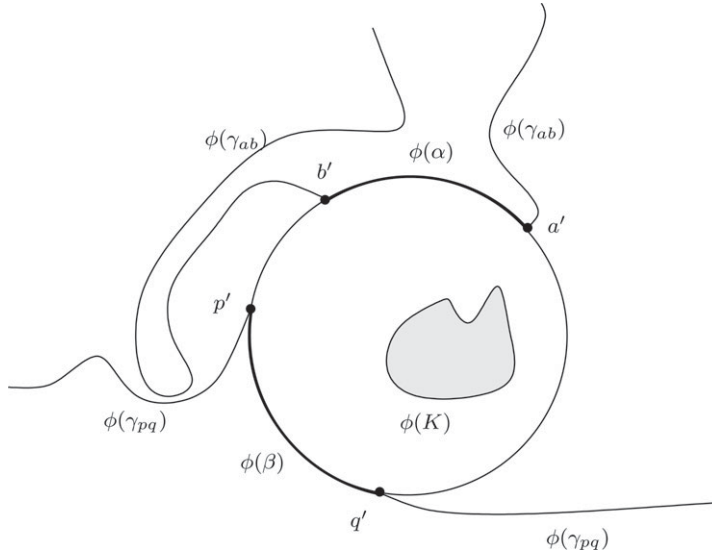
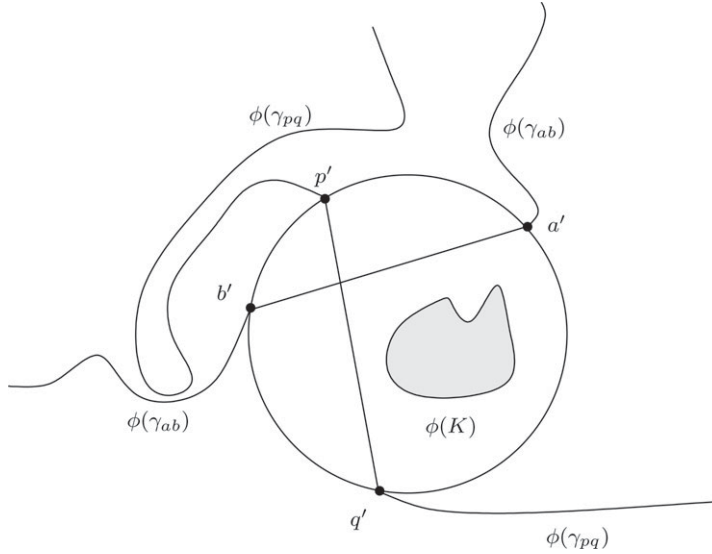
$$\begin{aligned} a' &= \phi \circ \gamma_{ab}(t_1) = Re^{i\theta_{a'}} \quad \text{and} \quad b' = \phi \circ \gamma_{ab}(t_2) = Re^{i\theta_{b'}}, \\ p' &= \phi \circ \gamma_{pq}(s_1) = Re^{i\theta_{p'}} \quad \text{and} \quad q' = \phi \circ \gamma_{pq}(s_2) = Re^{i\theta_{q'}}, \end{aligned}$$

where $\theta_{a'}, \theta_{b'}, \theta_{p'}, \theta_{q'} \in [0, 2\pi)$. Note that a', b', p', q' must be distinct points on the circle $C = \{z \in \mathbb{R}^2 : |z| = R\}$, for otherwise $\text{im } \gamma_{ab}$ and $\text{im } \gamma_{pq}$ are disjoint and K is empty. Thus we may consider the cyclic order of a', b', p', q' on the circle C . First suppose that a' and b' are adjacent in this order (Figure 4). Without loss of generality, we may assume that $\theta_{a'} < \theta_{b'} < \theta_{p'} < \theta_{q'}$. Note that a', b', p', q' must be distinct points on the circle $C = \{z \in \mathbb{R}^2 : |z| = R\}$, for otherwise $\text{im } \gamma_{ab}$ and $\text{im } \gamma_{pq}$ are disjoint and K is empty. Thus we may consider the cyclic order of a', b', p', q' on the circle C . First suppose that a' and b' are adjacent in this order. Without loss of generality, we may assume that $\theta_{a'} < \theta_{b'} < \theta_{p'} < \theta_{q'}$. Define arcs α and β in X by

$$\alpha = \phi^{-1}(\{Re^{i\theta} \in \mathbb{C} : \theta_{a'} \leq \theta \leq \theta_{b'}\}) \quad \text{and} \quad \beta = \phi^{-1}(\{Re^{i\theta} \in \mathbb{C} : \theta_{p'} \leq \theta \leq \theta_{q'}\}),$$

and set

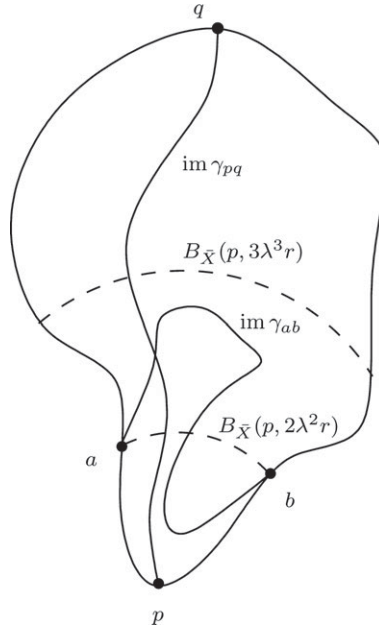
$$\gamma'_{ab} = \text{im}(\gamma_{ab}|_{[0,t_1]}) \cup \alpha \cup \text{im}(\gamma_{ab}|_{(t_2,1]}) \quad \text{and} \quad \gamma'_{pq} = \text{im}(\gamma_{pq}|_{[0,s_1]}) \cup \beta \cup \text{im}(\gamma_{pq}|_{(s_2,1]}).$$

FIGURE 4. The case that a' and b' are adjacent.FIGURE 5. The case that a' and b' are not adjacent.

Then γ'_{ab} connects a to b without intersecting γ'_{pq} . However, Lemma 4.12 implies that the points a and b are in different components of $\bar{X} - \gamma'_{pq}$, yielding a contradiction.

Now suppose that a' and b' are not adjacent in the order on C (Figure 5). We may assume without loss of generality that $\theta_{a'} < \theta_{p'} < \theta_{b'} < \theta_{q'}$. In this case, define $\alpha \subseteq X$ to be the inverse image under ϕ of the line segment in \mathbb{C} from a' to b' . Similarly, let $\beta \subseteq X$ be the inverse image under ϕ of the line segment from p' to q' .

Define γ'_{ab} and γ'_{pq} as before. Then it is clear that p' and q' are in different components of $\bar{X} - \text{im } \gamma'_{ab}$, as the line segments from p' to q' and a' to b' have a single transversal intersection. Furthermore, p' can be connected to p without intersecting $\text{im } \gamma'_{ab}$, and similarly for q' and q . Thus p and q are in different components of $\bar{X} - \text{im } \gamma'_{ab}$. Lemma 4.12 then implies that they are in different components of $\bar{X} - \text{im } \gamma_{ab}$. \square

FIGURE 6. The boundary is LLC_1 .

PROPOSITION 4.14. *The boundary ∂X is LLC_1 with constant depending only on the LLC constant of X . In particular, ∂X is locally connected.*

Proof. Let $p \in \partial X$, and $r > 0$. It suffices to find a continuum E such that

$$B_{\bar{X}}(p, r) \cap \partial X \subseteq E \subseteq B_{\bar{X}}(p, 4\lambda^4 r) \cap \partial X.$$

By Proposition 4.7, we may assume that there is some point $q \in \partial X - B_{\bar{X}}(p, 4\lambda^4 r)$. The $\widetilde{\text{LLC}}$ condition provides a crosscut γ_{pq} connecting p to q . Let U and V be the components of $X - \text{im } \gamma_{pq}$, and set $A = \bar{U} \cap \partial X$ and $B = \bar{V} \cap \partial X$. By Lemma 4.10(iii), A and B are connected. As $\{p, q\} = A \cap B$ and $d(p, q) > 4\lambda^4 r$, we may find distinct points $a \in A$ and $b \in B$ such that $d(p, a) = 2\lambda^2 r$ and $d(p, b) = 2\lambda^2 r$. The $\lambda\text{-}\widetilde{\text{LLC}}_1$ condition provides a crosscut γ_{ab} connecting a to b with $\text{im } \gamma_{ab} \subseteq B_{\bar{X}}(p, 3\lambda^3 r)$ (Figure 6). Let W be the component of $X - \text{im } \gamma_{ab}$ with $p \in \bar{W} \cap \partial X$, and set $E := \bar{W} \cap \partial X$. Applying Lemma 4.10(iii) again, we see that the set E is a continuum.

We first show that

$$E \subseteq B_{\bar{X}}(p, 4\lambda^4 r) \cap \partial X.$$

Suppose that there is a point $x \in E - B_{\bar{X}}(p, 4\lambda^4 r)$. By the $\lambda\text{-}\widetilde{\text{LLC}}_2$ condition, there is a path connecting x to q without intersecting $B_{\bar{X}}(p, 4\lambda^3 r)$. This implies that p and q are in the same component of $\bar{X} - \gamma_{ab}$. By Lemma 4.10(ii), the points a and b lie in different components of $X - \text{im } \gamma_{pq}$. This contradicts Lemma 4.13.

We now show that $B_{\bar{X}}(p, r) \cap \partial X \subseteq E$. Since γ_{ab} is uniformly continuous, we may find parameters $0 < t_a < 1$ and $0 < t_b < 1$ such that

$$\text{diam}(\gamma_{ab}([0, t_a])) \leq \lambda^2 r \quad \text{and} \quad \text{diam}(\gamma_{ab}([t_b, 1])) \leq \lambda^2 r.$$

Set $a' = \gamma_{ab}(t_a)$ and $b' = \gamma_{ab}(t_b)$. Then $a', b' \in X - B_{\bar{X}}(p, \lambda^2 r)$, and so the $\lambda\text{-}\widetilde{\text{LLC}}_2$ condition provides an embedding $\gamma_{a', b'} : [0, 1] \rightarrow X$ such that

$$\gamma(0) = a', \quad \gamma(1) = b', \quad \text{and} \quad \text{im } \gamma \subseteq X - B_{\bar{X}}(p, \lambda r).$$

Consider that the set

$$S = \gamma_{ab}([0, t_a]) \cup \text{im } \gamma_{a', b'} \cup \gamma_{ab}([t_b, 1])$$

does not intersect $B_{\bar{X}}(p, \lambda r)$, and is the image of a path in \bar{X} . Since the image of a path in \bar{X} is arc-connected, we may find a crosscut γ' connecting a to b with $\text{im } \gamma' \subseteq S$. Furthermore, after re-parametrization, we may find a compact interval $I \subseteq (0, 1)$ such that $\gamma'(t) = \gamma_{ab}(t)$ for all $t \in [0, 1] - I$. Suppose that there is a point $x \in B_{\bar{X}}(p, r) \cap \partial X$ which is not contained in E . Then x and p are in different components of $\bar{X} - \text{im } \gamma_{ab}$. By Lemma 4.12, this implies that x and p are in different components $\bar{X} - \text{im } \gamma'$. However, the $\lambda\text{-}\text{LLC}_1$ condition shows that x and p may be connected by an arc contained in $B_{\bar{X}}(p, \lambda r)$. This is a contradiction. \square

PROPOSITION 4.15. *If Y is a $\lambda\text{-}\text{LLC}_1$ metric space homeomorphic to \mathbb{S}^1 , then Y is also $\lambda\text{-}\text{LLC}$.*

Proof. The statement will follow from two elementary topological facts about the circle \mathbb{S}^1 :

- if x, y, z, w are distinct points in \mathbb{S}^1 , then x and y are in different components of $\mathbb{S}^1 - \{z, w\}$ if and only if z and w are in different components of $\mathbb{S}^1 - \{x, y\}$;
- if $E \subseteq \mathbb{S}^1$ is a continuum containing points $z, w \in \mathbb{S}^1$, then E contains at least one of the components of $\mathbb{S}^1 - \{z, w\}$.

Let Y be a metric space homeomorphic to \mathbb{S}^1 and satisfying the $\lambda\text{-}\text{LLC}_1$ condition. We will verify the $\lambda\text{-}\text{LLC}_2$ condition. Let $z \in Y$, $r > 0$, and $x, y \in Y - B(z, r)$. We seek a continuum in $Y - B(z, r/\lambda)$ containing x and y . Let I_1 and I_2 be the components of $Y - \{x, y\}$. Without loss of generality, we may assume that $z \in I_1$. If I_2 does not intersect $B(z, r/\lambda)$, then the closure of I_2 is the desired continuum. Suppose that there is a point $w \in B(z, r/\lambda) \cap I_2$. Since Y satisfies the $\lambda\text{-}\text{LLC}_1$ condition, there is a continuum $E \subseteq B(z, r)$ containing z and w . Let J_1 and J_2 be components of $Y - \{z, w\}$. By the first fact mentioned above, we may assume that $x \in J_1$ and $y \in J_2$. By the second fact above, either J_1 or J_2 is contained in E . This contradicts the assumption that $x, y \in Y - B(z, r)$. \square

Proof of Theorem 1.3. Let X be a bounded, $\lambda\text{-}\text{LLC}$ metric space homeomorphic to the disk. By Propositions 4.7, 4.8, 4.11, and 4.14, the boundary ∂X is a locally connected metric continuum such that the removal of one point does not separate the space, whereas the removal of two does. A recognition theorem of point-set topology [21] states that such a space is homeomorphic to the circle \mathbb{S}^1 . Propositions 4.14 and 4.15 show that ∂X is $\lambda'\text{-}\text{LLC}$, where λ' depends only on λ .

Tukia and Väisälä [19] characterized metric spaces quasimetrically equivalent to \mathbb{S}^1 in the following way: if Y is a metric space homeomorphic to \mathbb{S}^1 , then Y is quasimetrically equivalent to \mathbb{S}^1 if and only if it is doubling and satisfies the LLC_1 condition. Furthermore, they show that the distortion function of the quasimetry can be chosen to depend only on the LLC_1 and doubling constants. This proves the final statement of Theorem 1.3.

5. The disk

Throughout this section, let (X, d) be a locally compact, bounded, and incomplete metric space. Let X' be the space obtained by gluing two copies of \bar{X} together by the identity map

along ∂X . We will denote elements of X' by $[x, i]$, where $x \in X$ and $i \in \{1, 2\}$; if $x \in \partial X$, then we will use the notation $[x, 1] = [x] = [x, 2]$. If $E \subseteq \bar{X}$, then we set $[E, i] := \{[x, i] : x \in E\}$. By local compactness, we have $\text{dist}(x, \partial X) > 0$ for each $x \in X$. There is a natural metric d' on the space X' given by

$$d'([x, i], [y, j]) := \begin{cases} d(x, y) & i = j, \\ \inf\{d(x, z) + d(z, y) : z \in \partial X\} & i \neq j. \end{cases}$$

Note that $\text{diam } X' \leq 2 \text{diam } \bar{X}$, and that X embeds isometrically in X' . If X is homeomorphic to \mathbb{D}^2 and ∂X is homeomorphic to \mathbb{S}^{2^1} , then X' is homeomorphic to \mathbb{S}^{2^2} .

REMARK 5.1. The triangle inequality shows that the projection map $[x, j] \mapsto x$ does not increase distance.

PROPOSITION 5.2. *Suppose that (X, d) is an Ahlfors Q -regular and LLC metric space such that ∂X contains at least two points. Then (X', d') is Ahlfors Q -regular and LLC, with data depending only on the data of (X, d) and the ratio $\text{diam } X / \text{diam } \partial X$.*

Proof. We begin by showing that (X', d') is Ahlfors Q -regular. By Proposition 2.10, we may assume that \bar{X} is Ahlfors Q -regular with constant K . Let $[a, i] \in X'$, and let $r \leq \text{diam } X'$. We first give a lower estimate for $\mathcal{H}_{X'}^Q(\bar{B}_{X'}([a, i], r))$. Note that $\frac{1}{2}r \leq \text{diam } \bar{X}$. Let $\epsilon > 0$, and consider any cover $\{\bar{B}_{X'}([x_n, i_n], r_n)\}$ of $\bar{B}_{X'}([a, i], r/2)$ by closed balls in X' of radius less than ϵ . Then $\{\bar{B}_{\bar{X}}(x_n, r_n)\}$ is a cover of $\bar{B}_{\bar{X}}(a, r/2)$ by closed balls in \bar{X} of radius less than ϵ . Thus

$$\frac{r^Q}{2^Q K} \leq \mathcal{H}_{\bar{X}}^Q(\bar{B}_{\bar{X}}(a, r/2)) \leq \mathcal{H}_{X'}^Q(\bar{B}_{X'}([a, i], r/2)) \leq \mathcal{H}_{X'}^Q(\bar{B}_{X'}([a, i], r)). \quad (5.1)$$

We now show an upper estimate. If $\{\bar{B}_{\bar{X}}(x_n, r_n)\}$ is any cover of $\bar{B}_{\bar{X}}(a, r)$ by closed balls in \bar{X} of radius less than ϵ , then

$$\{\bar{B}_{X'}([x_n, 1], r_n)\} \cup \{\bar{B}_{X'}([x_n, 2], r_n)\}$$

is a cover of $\bar{B}_{X'}([a], r)$ by closed balls in X' of radius less than ϵ . Therefore,

$$\mathcal{H}_{X'}^Q(\bar{B}_{X'}([a], r)) \leq 2Kr^Q. \quad (5.2)$$

Combining (5.1) and (5.2), we see that X' is Ahlfors Q -regular with constant $\max\{2^Q K, 2K\}$.

We now show that (X', d') is LLC. By Proposition 2.7, we may assume that X is λ -LLC. Let $[a, i] \in X'$ and $r > 0$. Let $[x, j]$ and $[y, k]$ be points in $B_{X'}([a, i], r)$. By Remark 5.1 we have $x, y \in B_{\bar{X}}(a, r)$.

First suppose that $a \in X$ and $r < \text{dist}(a, \partial X)$. This implies that $i = j = k$. The λ -LLC condition on \bar{X} provides a continuum $E_{xy} \subseteq B_{\bar{X}}(a, \lambda r)$ containing x and y . Then

$$[E_{xy}, i] \subseteq B_{X'}([a, i], \lambda r)$$

is a continuum in X' containing $[x, j]$ and $[y, k]$.

Next, suppose that $a \in \partial X$ and $r > 0$. The λ -LLC condition on \bar{X} provides continua E_{xa} and E_{ya} contained in $B_{\bar{X}}(a, \lambda r)$ and containing $\{x, a\}$ and $\{y, a\}$, respectively. Now

$$[E_{xa}, j] \cup [E_{ya}, k] \subseteq B_{X'}([a], \lambda r)$$

is a continuum in X' containing $[x, j]$ and $[y, k]$.

Finally we consider the case that $a \in X$ and $r \geq \text{dist}(a, \partial X)$. There is a point $a' \in \partial X$ such that $d(a, a') < 2r$. Consider that

$$B_{X'}([a, i], r) \subseteq B_{X'}([a'], 3r) \subseteq B_{X'}([a'], 3\lambda r) \subseteq B_{X'}([a, i], (3\lambda + 2)r).$$

These inclusions and the discussion above show that there is a continuum E containing $[x, j]$ and $[y, k]$ such that $E \subseteq B_{X'}([a, i], (3\lambda + 2)r)$. We have now shown that X' is $(3\lambda + 2)$ -LLC₁.

Next, we show the LLC₂ condition. Let $[x, j]$ and $[y, k]$ be points in $X' - B_{X'}([a, i], r)$. First suppose that $a \in X$ and $r < \text{dist}(a, \partial X)$. This implies that neither x nor y is in $B_{\bar{X}}(a, r)$. If a' is any point of ∂X , then the λ -LLC condition provides continua $E_{xa'}$ and $E_{ya'}$ contained in $\bar{X} - B_{\bar{X}}(a, r/\lambda)$ and containing $\{x, a'\}$ and $\{y, a'\}$, respectively. Then

$$[E_{xa'}, j] \cup [E_{ya'}, k] \subseteq X' - B_{X'}([a, i], r/\lambda)$$

is a continuum in X' containing $[x, j]$ and $[y, k]$.

Now suppose that $a \in \partial X$. Then it is required that $r \leq \text{diam } X' \leq 2 \text{diam } X$, for otherwise $X' - B_{X'}([a], r) = \emptyset$. Setting $\alpha = \text{diam } X / \text{diam } \partial X$, we have

$$\frac{r}{8\alpha} \leq \frac{\text{diam } \partial X}{4} < \frac{\text{diam } \partial X}{2}.$$

Thus there is a point $a' \in \partial X - B_{\bar{X}}(a, r/8\alpha)$. Furthermore, neither x nor y is in $B_{\bar{X}}(a, r/8\alpha)$. Thus the λ -LLC₂ condition on \bar{X} provides continua $E_{xa'}$ and $E_{ya'}$ contained in $\bar{X} - B_{\bar{X}}(a, r/8\alpha\lambda)$ and containing $\{x, a'\}$ and $\{y, a'\}$, respectively. Now,

$$[E_{xa'}, j] \cup [E_{ya'}, k] \subseteq X' - B_{X'}([a, i], r/8\alpha\lambda)$$

is a continuum in X' containing $[x, j]$ and $[y, k]$.

We return to the case $a \in X$, and now allow that $r < 64\alpha\lambda \text{dist}(a, \partial X)$. Then $[x, j]$ and $[y, k]$ are in the complement of $B_{X'}([a, i], r/64\alpha\lambda)$. The first case above provides a continuum $E \subseteq X' - B_{X'}([a, i], r/64\alpha\lambda^2)$ containing $[x, j]$ and $[y, k]$. If $r \geq 64\alpha\lambda \text{dist}(a, \partial X)$, then we may find a point $a' \in \partial X$ such that $d(a, a') < r/(32\alpha\lambda)$. This implies that $[x, j]$ and $[y, k]$ are in the complement of $B_{X'}([a'], r/2)$. Now, the second case above provides a continuum E containing $[x, j]$ and $[y, k]$ such that

$$E \subseteq X' - B_{X'}\left([a'], \frac{r}{16\alpha\lambda}\right) \subseteq X' - B_{X'}\left([a, i], \frac{r}{32\alpha\lambda}\right).$$

We have now shown that X' is $64\alpha\lambda^2$ -LLC₂. □

EXAMPLE 5.3. In the above proposition, the Ahlfors 2-regularity and LLC₁ constants of (X', d') do not depend on α . However, the dependence of the LLC₂ constant of (X', d') on this ratio cannot be avoided. Let $\epsilon > 0$ and consider $X_\epsilon = \mathbb{S}^2 - \bar{B}(a, \epsilon)$. The LLC₂ constant of X_ϵ does not depend on ϵ , while the LLC₂ constant of X'_ϵ tends to infinity as ϵ tends to zero. The spaces X_ϵ also show that the distortion function of the uniformizing quasismorphism provided by Theorem 1.2 depends on α as well. To see this, suppose that $f_\epsilon : X_\epsilon \rightarrow \mathbb{D}^2$ is an η -quasismorphism homeomorphism. The map f_ϵ extends to an η -quasismorphism homeomorphism $\tilde{f}_\epsilon : \bar{X}_\epsilon \rightarrow \mathbb{D}^2$ sending ∂X_ϵ to $\partial \mathbb{D}^2$. For sufficiently small ϵ , Proposition 2.1 shows that

$$1 = \frac{\text{diam } \partial \mathbb{D}^2}{\text{diam } \mathbb{D}^2} \leq \eta \left(\frac{2 \text{diam } \partial X_\epsilon}{\text{diam } \bar{X}_\epsilon} \right) = \eta(2\epsilon).$$

Letting ϵ tend to zero yields a contradiction, since $\eta(t) \rightarrow 0$ as $t \rightarrow 0$.

Proof of Theorem 1.2(iii). Suppose that X is an Ahlfors 2-regular, LLC, and bounded metric space homeomorphic to the plane, with $\text{card } \partial X \geq 2$. Remark 4.1 shows that we may apply Theorem 1.3 and conclude that the boundary ∂X is homeomorphic to the circle. This implies that the doubled space X' is homeomorphic to \mathbb{S}^2 . Theorem 5.2 shows that X' is Ahlfors 2-regular and LLC, with data depending only on the data of X and the ratio $\text{diam } X / \text{diam } \partial X$. Theorem 1.1 provides a quasismorphism homeomorphism $f : X' \rightarrow \mathbb{S}^2$ with distortion function depending only on the data of X' . Since X embeds isometrically into X' , Theorem 2.4 shows

that $f(X)$ is an LLC disk inside \mathbb{S}^2 , and Proposition 2.1(iii) shows that $\partial f(X)$ is a continuum. Theorem 2.6 provides a quasimetric homeomorphism $g : f(X) \rightarrow \mathbb{D}^2$ with the distortion function depending only on the LLC-constant of $f(X)$, and hence only on the data of X and the ratio $\text{diam } X / \text{diam } \partial X$. The map $g \circ f$ is the desired quasimetric homeomorphism.

6. The plane and the half-plane

Throughout this section, let (X, d) be a connected and unbounded metric space. We wish to ‘warp’ (X, d) to create a bounded metric space. This warping process, which was also employed in [5], is analogous to obtaining the standard extrinsic metric on \mathbb{S}^2 from the standard metric on \mathbb{R}^2 .

Fix a basepoint $p \in X$, and define for all $x, y \in X$

$$\rho_p(x, y) := \frac{d(x, y)}{(1 + d(x, p))(1 + d(y, p))}.$$

In general, ρ_p is not a metric on X . To force the triangle inequality, we define

$$\widehat{d}_p(x, y) = \inf \sum_{i=0}^{k-1} \rho_p(x_i, x_{i+1}),$$

where the infimum is taken over all finite sequences of points $x = x_0, x_1, \dots, x_k = y$ in X .

As $p \in X$ will remain fixed throughout, we will suppress the reference to p in the definitions above, using instead $\widehat{d} = \widehat{d}_p$ and $\rho = \rho_p$. For further ease of notation, for all $x \in X$ we set

$$h(x) := \frac{1}{1 + d(x, p)}.$$

REMARK 6.1. Note that for any $u, v \in X$ we have

$$|h(u) - h(v)| = h(u)h(v) |d(v, p) - d(u, p)| \leq \rho(u, v).$$

This and the triangle inequality show that for any sequence $x = x_0, x_1, \dots, x_k = y$ of points in X , we have

$$\sum_{i=0}^{k-1} \rho(x_i, x_{i+1}) \geq |h(x) - h(y)|.$$

Thus for all points x and y in X we have

$$\widehat{d}(x, y) \geq |h(x) - h(y)|. \quad (6.1)$$

In order to show that \widehat{d} is a metric on X , we need the following lemma which is proven in [5].

LEMMA 6.2. For all $x, y \in X$, we have

$$\frac{1}{4}\rho(x, y) \leq \widehat{d}(x, y) \leq \rho(x, y).$$

Lemma 6.2 shows that $\widehat{d}(x, y) = 0$ implies $x = y$ for all points x and y in X . It follows from the definitions that \widehat{d} is symmetric and satisfies the triangle inequality. Thus, we define the ‘warped’ version of (X, d) to be the metric space (X, \widehat{d}) . In this warped space, distances from p may be calculated from the d -distance from p .

LEMMA 6.3. If $x \in X$, then $\widehat{d}(x, p) = 1 - h(x)$.

Proof. From Lemma 6.2, we see that $\widehat{d}(x, p) \leq \rho(x, p) = 1 - h(x)$. On the other hand, setting $y = p$ in inequality (6.1) shows that $\widehat{d}(x, p) \geq |h(x) - 1| = 1 - h(x)$. \square

Let \widehat{X} denote the completion of (X, \widehat{d}) , and let $\widehat{\partial}X = \widehat{X} - X$ be the metric boundary of (X, \widehat{d}) . We seek a description of $\widehat{\partial}X$ in terms of ∂X .

LEMMA 6.4. *Let $\{x_n\}$ be a sequence in X . If $d(x_n, p) \rightarrow \infty$, then $\{x_n\}$ is a non-convergent \widehat{d} -Cauchy sequence. Conversely, if $\{x_n\}$ is a \widehat{d} -Cauchy sequence that is d -unbounded, then $d(x_n, p) \rightarrow \infty$.*

Proof. Suppose that $\{x_n\} \subseteq X$ satisfies $d(x_n, p) \rightarrow \infty$, and let $\epsilon > 0$. We may find some integer $N > 0$ such that if $n \geq N$, then

$$\frac{1}{1 + d(x_n, p)} < \frac{\epsilon}{2}.$$

For $n \geq N$ and k any positive integer, Lemma 6.2 and the triangle inequality show that

$$\widehat{d}(x_n, x_{n+k}) \leq \frac{d(x_n, p) + d(x_{n+k}, p)}{(1 + d(x_n, p))(1 + d(x_{n+k}, p))} \leq \frac{1}{1 + d(x_{n+k}, p)} + \frac{1}{1 + d(x_n, p)} < \epsilon.$$

This shows that $\{x_n\}$ is a \widehat{d} -Cauchy sequence. Suppose that there is some $x \in X$ such that $\widehat{d}(x_n, x) \rightarrow 0$. Then by Lemma 6.3

$$1 - h(x) = \widehat{d}(x, p) = \lim_{n \rightarrow \infty} \widehat{d}(x_n, p) = 1 - \lim_{n \rightarrow \infty} h(x_n) = 1.$$

This implies that $h(x) = 0$, which is impossible. Thus $\{x_n\}$ does not converge to a point in X .

Now, let $\{x_n\} \subseteq X$ be a \widehat{d} -Cauchy sequence that is d -unbounded. If $d(x_n, p)$ does not tend to infinity, then there exists some $R \geq 0$ such that for infinitely many positive integers n we have $d(x_n, p) < R$. By Lemma 6.3

$$\widehat{d}(x_n, p) = \frac{d(x_n, p)}{1 + d(x_n, p)},$$

and so there are infinitely many positive integers n such that $x_n \in B_{\widehat{d}}(p, R/(1 + R))$. On the other hand, $\{x_n\}$ is d -unbounded, and so there are infinitely many positive integers n such that $d(x_n, p) > 2R$. As a result, there are infinitely many positive integers n such that

$$x_n \notin B_{\widehat{d}}(p, 2R/(1 + 2R)).$$

This is a contradiction. \square

REMARK 6.5. If $\{x_n\}$ and $\{y_n\}$ are \widehat{d} -Cauchy sequences which are d -unbounded, then by Lemma 6.4 the sequence $\{x_1, y_1, x_2, y_2, \dots\}$ has no d -bounded subsequences and is again a \widehat{d} -Cauchy sequence. Thus we may define a distinguished point $\infty \in \widehat{\partial}X$ corresponding to \widehat{d} -Cauchy sequences which are d -unbounded.

There is a special relationship between the basepoint p and the point $\infty \in \widehat{\partial}X$.

LEMMA 6.6. *If $x \in \bar{X}$, then $\widehat{d}(x, \infty) = h(x)$.*

Proof. By the definition of completion, it suffices to prove the result in the case that $x \in X$. We must show that if $\{y_n\} \subseteq X$ is any \widehat{d} -Cauchy sequence which is d -unbounded, then

$\widehat{d}(x, y_n) \rightarrow h(x)$ as $n \rightarrow \infty$. Lemma 6.2 shows that

$$\lim_{n \rightarrow \infty} \widehat{d}(x, y_n) \leq \lim_{n \rightarrow \infty} \rho(x, y_n) = h(x).$$

By Remark 6.1 we have, for each $n \in \mathbb{N}$,

$$\widehat{d}(x, y_n) \geq |h(x) - h(y_n)|.$$

By Lemma 6.4, $\{y_n\}$ satisfies $d(y_n, p) \rightarrow \infty$, and so $h(y_n)$ tends to zero. Thus

$$\lim_{n \rightarrow \infty} \widehat{d}(x, y_n) \geq h(x). \quad \square$$

REMARK 6.7. Since $h(x) = 1/(1 + d(x, p))$, Lemmas 6.3 and 6.6 provide the first step in relating the metrics d and \widehat{d} . An example of their usefulness is the equality

$$B_{\widehat{d}}(\infty, r) = \begin{cases} \widehat{X}, & r > 1, \\ \widehat{X} - \{p\}, & r = 1, \\ \widehat{X} - \bar{B}_d(p, \frac{1-r}{r}), & r < 1. \end{cases}$$

In particular, this shows that $\text{diam}(\widehat{X}, \widehat{d}) \leq 2$. It is also convenient to record that for all $r > 0$,

$$X - B_d(p, r) = \bar{B}_{\widehat{d}}\left(\infty, \frac{1}{1+r}\right) \cap X.$$

It is not possible to give such an exact description of every \widehat{d} -ball, but the following lemma shows that the metrics d and \widehat{d} are ‘comparable away from infinity’. This fact is the essential ingredient in showing that if (X, d) is Ahlfors Q -regular, then so is (X, \widehat{d}) .

LEMMA 6.8. Let $C > 1$. If $a \in X$, and $r \leq \widehat{d}(a, \infty)/C$, then

$$B_{X,d}\left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{C}{C+1}\right) \subseteq B_{X,\widehat{d}}(a, r) \subseteq B_{X,d}\left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{4C}{C-1}\right) \quad (6.2)$$

Furthermore, if $R \leq 1/(\widehat{d}(a, \infty)(C+1))$, then

$$B_{X,\widehat{d}}\left(a, R\widehat{d}(a, \infty)^2 \cdot \frac{C-1}{4C}\right) \subseteq B_{X,d}(a, R) \subseteq B_{X,\widehat{d}}\left(a, R\widehat{d}(a, \infty)^2 \cdot \frac{C+1}{C}\right). \quad (6.3)$$

The inclusions (6.2) and (6.3) also hold when all open balls are replaced with closed balls.

Proof. Let $x \in X$. By Lemma 6.6 we have

$$d(a, x) = \frac{\rho(a, x)}{\widehat{d}(a, \infty)\widehat{d}(x, \infty)}.$$

Lemma 6.2 and the triangle inequality imply that

$$\frac{\widehat{d}(a, x)}{\widehat{d}(a, \infty)(\widehat{d}(a, \infty) + \widehat{d}(a, x))} \leq d(a, x) \leq \frac{4\widehat{d}(a, x)}{\widehat{d}(a, \infty)(\widehat{d}(a, \infty) - \widehat{d}(a, x))}. \quad (6.4)$$

Now suppose that $r \leq \widehat{d}(a, \infty)/C$. If $x \in B_{X,\widehat{d}}(a, r)$, the second inequality in (6.4) yields

$$d(a, x) < \left(\frac{4C}{C-1}\right) \frac{\widehat{d}(a, x)}{\widehat{d}(a, \infty)^2}.$$

This yields the second inclusion in (6.2). If

$$x \in B_{X,d} \left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{C}{C+1} \right), \quad (6.5)$$

then the assumption that $r \leq \widehat{d}(a, \infty)/C$ and the first inequality in (6.4) imply that

$$\frac{\widehat{d}(a, x)}{\widehat{d}(a, \infty)(\widehat{d}(a, \infty) + \widehat{d}(a, x))} \leq d(a, x) \leq \frac{1}{\widehat{d}(a, \infty)(C+1)}.$$

This implies that $C\widehat{d}(a, x) \leq \widehat{d}(a, \infty)$. Using this estimate, (6.4) and (6.5) now yield

$$\left(\frac{C}{C+1} \right) \frac{\widehat{d}(a, x)}{\widehat{d}(a, \infty)^2} \leq d(a, x) < \frac{r}{\widehat{d}(a, \infty)^2} \left(\frac{C}{C+1} \right).$$

The first inclusion in (6.2) follows.

The inclusions (6.3) follow from (6.2). Note that if $R \leq 1/(\widehat{d}(a, \infty)(C+1))$, then

$$\max \left(R\widehat{d}(a, \infty)^2 \cdot \frac{C-1}{4C}, R\widehat{d}(a, \infty)^2 \cdot \frac{C+1}{C} \right) \leq \frac{\widehat{d}(a, \infty)}{C}.$$

Thus we may apply (6.2) with

$$r = R\widehat{d}(a, \infty)^2 \cdot \frac{C-1}{4C} \quad \text{and} \quad r = R\widehat{d}(a, \infty)^2 \cdot \frac{C+1}{C}.$$

This yields (6.3).

The proof of the statement where all open balls are replaced with closed balls is identical. \square

LEMMA 6.9. *Let $\{x_n\} \subseteq X$ be a d -bounded sequence. Then $\{x_n\}$ is a d -Cauchy sequence if and only if it is a \widehat{d} -Cauchy sequence. Furthermore $\{x_n\}$ d -converges to a point $x \in X$ if and only if it \widehat{d} -converges to x .*

Proof. By Lemma 6.2, we see that $\widehat{d}(x, y) \leq d(x, y)$ for all points x and y in X . This implies that if $\{x_n\}$ is a d -Cauchy sequence, then it is a \widehat{d} -Cauchy sequence. Furthermore, this shows that if $\{x_n\}$ d -converges to a point $x \in X$, then it \widehat{d} -converges to x as well.

Now suppose that $\{x_n\}$ is a \widehat{d} -Cauchy sequence which is d -bounded. There is some $R > 0$ such that $\{x_n\} \subseteq B_d(p, R)$. By Lemma 6.6, this implies that for all $n \in \mathbb{N}$,

$$\frac{1}{2(1+R)} < \frac{\widehat{d}(x_n, \infty)}{2}. \quad (6.6)$$

Let $\epsilon > 0$. As $\{x_n\}$ is a \widehat{d} -Cauchy sequence, there is some N such that for all $n \geq N$ we have $x_n \in B_{\widehat{d}}(x_N, \epsilon')$, where

$$\epsilon' := \min \left(\frac{\epsilon}{8(1+R)^2}, \frac{1}{2(1+R)} \right).$$

Inequality (6.6) shows that we may apply the inclusion (6.2) to $B_{\widehat{d}}(x_N, \epsilon')$ with constant $C = 2$. Thus we see that for all $n \geq N$

$$x_n \in B_d \left(x_N, \frac{8\epsilon'}{\widehat{d}(x_N, \infty)^2} \right) \subseteq B_d(x_N, \epsilon).$$

This shows that $\{x_n\}$ is a d -Cauchy sequence and that if $\{x_n\}$ \widehat{d} -converges to a point $x \in X$, then it d -converges to x as well. \square

PROPOSITION 6.10. *There is a bijection between $\widehat{\partial}X$ and $\partial X \cup \{\infty\}$.*

Proof. By Remark 6.5 and Lemma 6.9, it suffices to show that if $\{x_n\}$ and $\{y_n\}$ are d -bounded sequences in X , then the sequence $\{x_1, y_1, x_2, y_2, \dots\}$ is a d -Cauchy sequence if and only if it is a \widehat{d} -Cauchy sequence. However, this also follows from Lemma 6.9, as $\{x_1, y_1, x_2, y_2, \dots\}$ is also d -bounded. \square

LEMMA 6.11. *The identity map $\iota : (X, d) \rightarrow (X, \widehat{d})$ is a θ -quasi-Möbius homeomorphism with $\theta(t) = 16t$.*

Proof. Clearly ι is a bijection; that it and its inverse are continuous follows from Lemma 6.9. It follows from Lemma 6.2 that ι is a $16t$ -quasi-Möbius map. \square

We now make rigorous the statement that the warping process is analogous to obtaining the extrinsic metric on \mathbb{S}^2 from the standard metric on \mathbb{R}^2 . For $n = 2, 3$, let $|\cdot|_{\mathbb{R}^n}$ denote the standard metric structure on \mathbb{R}^n , and let $|\cdot|_{\mathbb{R}^n}$ denote the corresponding warped metric with basepoint at the origin.

LEMMA 6.12. *The space $(\mathbb{R}^2, |\cdot|_{\mathbb{R}^2})$ is bi-Lipschitz equivalent to the extrinsic metric on \mathbb{S}^{2*} .*

Proof. Let $s : \mathbb{R}^2 \rightarrow \mathbb{S}^2 - \{0, 0, 1\}$ be the stereographic projection map. By Lemma 6.2, it suffices to show that there is a constant $L > 1$ such that for all $x, y \in \mathbb{R}^2$

$$\frac{1}{L} \frac{|x - y|_{\mathbb{R}^2}}{(1 + |x|)(1 + |y|)} \leq |s(x) - s(y)|_{\mathbb{R}^3} \leq L \frac{|x - y|_{\mathbb{R}^2}}{(1 + |x|)(1 + |y|)}.$$

A calculation shows that

$$|s(x) - s(y)|_{\mathbb{R}^3} = \frac{2|x - y|_{\mathbb{R}^2}}{\sqrt{(1 + |x|^2)(1 + |y|^2)}},$$

and the result follows with $L = 4$. \square

We now have the tools needed to prove that the warping procedure preserves Ahlfors Q -regularity and the LLC condition quantitatively.

PROPOSITION 6.13. *Let (X, d) be a connected and unbounded metric space, and let $Q > 0$. If (X, d) is Ahlfors Q -regular with constant K , then (X, \widehat{d}) is Ahlfors Q -regular with a constant depending only on Q and K .*

Proof. Throughout this proof, we will only consider balls centered in X as objects in X , not in the completion \widehat{X} . Thus we will use the notation

$$\bar{B}_d(a, r) = \bar{B}_{X,d}(a, r) \quad \text{and} \quad \bar{B}_{\widehat{d}}(a, r) = \bar{B}_{X,\widehat{d}}(a, r).$$

The general method is to construct a cover of a ball in one metric from a cover of a ball in the other metric, in a quantitative way. The main tool is Lemma 6.8. For technical reasons which will become clear later, we fix $\widetilde{C} = 2$ and set

$$C := \frac{8\widetilde{C}(\widetilde{C} + 1)}{\widetilde{C} - 1} > 2.$$

Let $a \in X$ and $r \leq \text{diam}(X, \widehat{d})$. The first step is to estimate $\mathcal{H}_d^Q(\bar{B}_{\widehat{d}}(a, r))$ in the case that $r \leq \widehat{d}(a, \infty)/C$. Since $C > 2$, we may fix $\epsilon > 0$ such that

$$\epsilon < \min \left\{ \frac{\widehat{d}(a, \infty) - 2r}{C}, r \right\}.$$

Suppose that $\bar{B}_{\widehat{d}}(a, r)$ is covered by a collection of closed balls $\{\widehat{B}_i\}_{i \in I}$, where $\widehat{B}_i := \bar{B}_{\widehat{d}}(x_i, r_i)$, $r_i < \epsilon$, and $\widehat{d}(a, x_i) < r + \epsilon$ for each $i \in I$. From this cover, we will construct a cover of a d -ball of radius roughly $r/\widehat{d}(a, \infty)^2$ by d -balls of radius roughly $r_i/\widehat{d}(a, \infty)^2$. Since the same factor appears in both the covered and covering balls, the resulting bounds on the Hausdorff measure of $\bar{B}_{X, \widehat{d}}(a, r)$ will be independent of a .

An application of (6.2) shows that

$$\bar{B}_d \left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{C}{C+1} \right) \subseteq \bar{B}_{\widehat{d}}(a, r).$$

We may also apply (6.2) to each \widehat{B}_i , because

$$r_i < \frac{\widehat{d}(a, \infty) - 2r}{C} \leq \frac{\widehat{d}(a, x_i) - 2r + \widehat{d}(x_i, \infty)}{C} \leq \frac{\widehat{d}(x_i, \infty)}{C}.$$

Accordingly,

$$\widehat{B}_i \subseteq \bar{B}_d \left(x_i, \frac{r_i}{\widehat{d}(x_i, \infty)^2} \cdot \frac{4C}{C-1} \right).$$

We also know that

$$\widehat{d}(x_i, \infty)^2 \geq (\widehat{d}(a, \infty) - \widehat{d}(a, x_i))^2 \geq (\widehat{d}(a, \infty) - r - \epsilon)^2 \geq \widehat{d}(a, \infty)^2 \left(\frac{C-1}{C} - \frac{\epsilon}{\widehat{d}(a, \infty)} \right)^2 > 0,$$

and hence

$$\widehat{B}_i \subseteq B_i := \bar{B}_d \left(x_i, \frac{r_i}{\widehat{d}(a, \infty)^2} \cdot \frac{4C}{C-1} \left(\frac{C-1}{C} - \frac{\epsilon}{\widehat{d}(a, \infty)} \right)^{-2} \right).$$

Note that the radius of each B_i is bounded above by a constant which is independent of $i \in I$ and tends to zero as ϵ tends to zero. Furthermore the collection $\{B_i\}_{i \in I}$ covers the ball

$$\bar{B}_d \left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{C}{C+1} \right).$$

This implies that

$$\mathcal{H}_d^Q \left(\bar{B}_d \left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{C}{C+1} \right) \right) \leq \frac{4C^3}{(C-1)^3 \widehat{d}(a, \infty)^2} \mathcal{H}_d^Q(\bar{B}_{\widehat{d}}(a, r)).$$

The Ahlfors Q -regularity of (X, d) now yields

$$\left(\frac{(C-1)^3}{4C^2(C+1)} \right)^Q \frac{r^Q}{K} \leq \mathcal{H}_d^Q(\bar{B}_{X, \widehat{d}}(a, r)). \quad (6.7)$$

To construct an upper bound for $\mathcal{H}_d^Q(\bar{B}_{\widehat{d}}(a, r))$, we fix a new $\epsilon > 0$ such that

$$\epsilon < \min \left\{ \frac{1}{2\widehat{d}(a, \infty)(\widetilde{C}+1)}, \frac{1}{\widetilde{C}+1} \right\}.$$

Consider a cover of

$$\bar{B}_d \left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{4\tilde{C}}{\tilde{C} - 1} \right)$$

by a collection of balls $\{B_i\}_{i \in I}$, where for each $i \in I$, $B_i := \bar{B}_d(x_i, r_i)$, $r_i < \epsilon$, and

$$x_i \in B_d \left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{4\tilde{C}}{\tilde{C} - 1} + \epsilon \right).$$

We will use $\{B_i\}$ to construct a cover of $\bar{B}_{\widehat{d}}(a, r)$ by \widehat{d} -balls of radius roughly $r_i \widehat{d}(a, \infty)^2$. This time, the cancellation will come from the fact that the d -ball we begin with has radius roughly $r/\widehat{d}(a, \infty)^2$.

As $C > \tilde{C}$, we have $r \leq \widehat{d}(a, \infty)/\tilde{C}$ as well, and so (6.2) implies

$$\bar{B}_{\widehat{d}}(a, r) \subseteq \bar{B}_d \left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{4\tilde{C}}{\tilde{C} - 1} \right).$$

We wish to apply (6.3) to each B_i , using the constant \tilde{C} . Note that for every point $z \in X$, $\widehat{d}(z, \infty) \leq 1$, and so the requirement that $r_i \leq (\widehat{d}(x_i, \infty)(\tilde{C} + 1))^{-1}$ is satisfied because $\epsilon < 1/(\tilde{C} + 1)$. Accordingly, for each $i \in I$,

$$B_i \subseteq \bar{B}_{\widehat{d}} \left(x_i, r_i \widehat{d}(x_i, \infty)^2 \cdot \frac{\tilde{C} + 1}{\tilde{C}} \right). \quad (6.8)$$

We now wish to estimate $\widehat{d}(x_i, \infty)^2$ independently of $i \in I$. We have assumed that $r \leq \widehat{d}(a, \infty)/C$, and so

$$r \leq \frac{\widehat{d}(a, \infty)(\tilde{C} - 1)}{8\tilde{C}(\tilde{C} + 1)}. \quad (6.9)$$

The upper bound on ϵ and (6.9) are exactly what is needed to show that

$$\frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{4\tilde{C}}{\tilde{C} - 1} + \epsilon \leq \frac{1}{\widehat{d}(a, \infty)(\tilde{C} + 1)}.$$

Invoking (6.3), we have

$$x_i \in B_d \left(a, \frac{r}{\widehat{d}(a, \infty)^2} \cdot \frac{4\tilde{C}}{\tilde{C} - 1} + \epsilon \right) \subseteq B_{\widehat{d}} \left(a, r \cdot \frac{4(\tilde{C} + 1)}{(\tilde{C} - 1)} + \epsilon \widehat{d}(a, \infty)^2 \cdot \frac{(\tilde{C} + 1)}{\tilde{C}} \right). \quad (6.10)$$

Now (6.9) and (6.10) provide the estimate

$$\widehat{d}(x_i, \infty)^2 \leq (\widehat{d}(a, x_i) + \widehat{d}(a, \infty))^2 \leq \widehat{d}(a, \infty)^2 \left(\frac{2\tilde{C} + 1}{2\tilde{C}} + \frac{\epsilon(\tilde{C} + 1)}{\tilde{C}} \right)^2.$$

Substituting this into (6.8), we have

$$B_i \subseteq \widehat{B}_i := \bar{B}_{\widehat{d}} \left(x_i, r_i \widehat{d}(a, \infty)^2 \left(\frac{2\tilde{C} + 1}{2\tilde{C}} + \frac{\epsilon(\tilde{C} + 1)}{\tilde{C}} \right)^2 \frac{(\tilde{C} + 1)}{\tilde{C}} \right).$$

Note that the radius of each \widehat{B}_i is bounded above by a constant which is independent of $i \in I$ and tends to zero as ϵ tends to zero. Moreover, the collection $\{\widehat{B}_i\}_{i \in I}$ covers the ball $\bar{B}_{\widehat{d}}(a, r)$. This combined with the Ahlfors Q -regularity of (X, d) and (6.7) shows that for $r \leq \widehat{d}(a, \infty)/C$,

$$\left(\frac{(C - 1)^3}{4C^2(C + 1)} \right)^Q \frac{r^Q}{K} \leq \mathcal{H}_{\widehat{d}}^Q(\bar{B}_{\widehat{d}}(a, r)) \leq K \left(\frac{(2\tilde{C} + 1)^2(\tilde{C} + 1)}{\tilde{C}^2(\tilde{C} - 1)} \right)^Q r^Q. \quad (6.11)$$

In order to estimate $\mathcal{H}^Q(\bar{B}_{\hat{d}}(a, r))$ in the case that $r > \hat{d}(a, \infty)/C$, we first estimate the Hausdorff Q -measure of balls centered at ∞ . Consider $\bar{B}_{\hat{X}, \hat{d}}(\infty, r) \cap X$, where $r \leq \text{diam}(\hat{X}, \hat{d}) \leq 2$. For $m \in \mathbb{N}$, define the half-open annulus

$$A_m := B_d\left(p, \frac{2^{m+1}}{r} - 1\right) - B_d\left(p, \frac{2^{m+1}}{r} - 1\right).$$

By Remark 6.7, we also have

$$A_m = \left(\bar{B}_{\hat{X}, \hat{d}}\left(\infty, \frac{r}{2^m}\right) - \bar{B}_{\hat{X}, \hat{d}}\left(\infty, \frac{r}{2^{m+1}}\right)\right) \cap X,$$

showing that $\bar{B}_{\hat{X}, \hat{d}}(\infty, r) \cap X = \bigcup_{m=0}^{\infty} A_m$. As the collection $\{A_m\}_{m \in \mathbb{N}}$ is disjoint, it is sufficient to find suitable bounds on the Q -Hausdorff measure of each A_m . Lemma 6.8 and a covering argument similar to those above yield that

$$\mathcal{H}_d^Q(A_m) \leq K \left(\frac{\tilde{C}+1}{\tilde{C}}\right)^Q \left(\frac{r}{2^{m+1}}\right)^Q.$$

Summing over $m \in \mathbb{N}$, we may conclude that

$$\mathcal{H}_{\hat{X}, \hat{d}}^Q(\bar{B}_{\hat{X}, \hat{d}}(\infty, r) \cap X) \leq K \left(\frac{4(\tilde{C}+1)}{\tilde{C}}\right)^Q \frac{r^Q}{2^Q - 1}. \quad (6.12)$$

Note that (6.12) also holds for $r \geq \text{diam}(X, \hat{d})$.

To establish a lower bound, we first note that X is connected, $\hat{d}(p, \infty) = 1$, and $r \leq 2$, and therefore there exists a point $a \in X$ such that $\hat{d}(a, p) = 1 - r/2$. Lemma 6.3 shows that $\hat{d}(a, \infty) = r/2$. Noting that

$$\bar{B}_{\hat{d}}(a, r/(2C)) \subseteq (\bar{B}_{\hat{X}, \hat{d}}(\infty, r) \cap X),$$

applying (6.11) to $\bar{B}_{\hat{d}}(a, r/(2C))$ shows that

$$\left(\frac{(C-1)^3}{8C^3(C+1)}\right)^Q \frac{r^Q}{K} \leq \mathcal{H}_{\hat{X}, \hat{d}}^Q(\bar{B}_{\hat{X}, \hat{d}}(\infty, r) \cap X). \quad (6.13)$$

Finally, we consider $\bar{B}_{\hat{d}}(a, r)$ with $a \in X$ and $r > \hat{d}(a, \infty)/C$. In this case,

$$\bar{B}_{\hat{d}}(a, r) \subseteq \bar{B}_{\hat{X}, \hat{d}}(\infty, (C+1)r) \cap X,$$

and so (6.12) provides the upper bound

$$\mathcal{H}_d^Q(\bar{B}_{\hat{d}}(a, r)) \leq K \left(\frac{4(\tilde{C}+1)^2}{\tilde{C}}\right)^Q \frac{r^Q}{2^Q - 1}. \quad (6.14)$$

For the lower bound, we first suppose that $r \leq C\hat{d}(a, \infty)$. Applying (6.7) to the $\bar{B}_{\hat{d}}(a, r/C^2)$ shows that

$$\left(\frac{(C-1)^3}{4C^4(C+1)}\right)^Q \frac{r^Q}{K} \leq \mathcal{H}_d^Q(\bar{B}_{\hat{d}}(a, r)).$$

If instead $r > C\hat{d}(a, \infty)$, then

$$\bar{B}_{\hat{d}}\left(\infty, \frac{C-1}{C}r\right) \cap X \subseteq \bar{B}_{\hat{d}}(a, r).$$

Thus we may apply (6.13) above to show that

$$\left(\frac{(C-1)^4}{8C^4(C+1)}\right)^Q \frac{r^Q}{K} \leq \mathcal{H}_d^Q(\bar{B}_{\hat{d}}(a, r)).$$

In either case,

$$\left(\frac{(C-1)^3}{8C^4(C+1)} \right)^Q \frac{r^Q}{K} \leq \mathcal{H}_d^Q(\bar{B}_{\hat{d}}(a, r)) \leq K \left(\frac{4(\tilde{C}+1)^2}{\tilde{C}} \right)^Q \frac{1}{2^Q-1} r^Q. \quad (6.15)$$

Recalling that $\tilde{C} = 2$, estimates (6.11) and (6.15) show that (X, \hat{d}) is Ahlfors Q -regular with constant depending only on K and Q . \square

PROPOSITION 6.14. *Let (X, d) be an unbounded and λ -LLC metric space. Then (X, \hat{d}) is λ' -LLC, where λ' depends only on λ .*

Proof. This follows immediately from Lemma 6.11 and Theorem 2.4. It can also be shown directly using Lemma 6.8. \square

Proof of Theorem 1.2(iv) and (v). Let (X, d) be an Ahlfors 2-regular, LLC metric space which is homeomorphic to \mathbb{R}^2 . Then by Remark 6.7 and Propositions 6.13 and 6.14, (X, \hat{d}) is a bounded, Ahlfors 2-regular, and LLC metric space with data depending only on the data of (X, d) .

If (X, d) is complete, then Lemma 6.10 shows that $\text{card } \partial X = 1$. Thus by Theorem 1.2(ii), there is an η -quasisymmetric homeomorphism $f : (X, \hat{d}) \rightarrow \mathbb{S}^{2*}$, where η depends only on the data of X . By Lemma 6.11 there is a quasi-Möbius homeomorphism $\iota : (X, d) \rightarrow (X, \hat{d})$. As the quasi-Möbius condition is invariant under composition with bi-Lipschitz maps, Lemmas 6.11 and 6.12 show that there is a quasi-Möbius homeomorphism $g : \mathbb{S}^{2*} \rightarrow \mathbb{R}^2$. The composition $g \circ f \circ \iota$ sends unbounded sequences to unbounded sequences, and so Theorem 2.2 shows that $g \circ f \circ \iota$ is quasisymmetric. As ι and g have fixed distortion functions, the distortion function of $g \circ f \circ \iota$ depends only on η and hence only on the data of X .

Now suppose that (X, d) is not complete. Lemma 6.10 shows that $\text{card } \partial X \geq 2$. In the construction of the warped space (X, \hat{d}) , the basepoint p can be chosen arbitrarily. As (\bar{X}, \hat{d}) is connected and unbounded, we may choose p such that there is a point $z \in \partial X$ such that $d(z, p) \leq 1$. Then $\hat{d}(\infty, z) \geq 1/2$, and so by Remark 6.7 and Lemma 6.10 we have

$$1 \leq \frac{\text{diam } \hat{X}}{\text{diam } \partial X} \leq 4.$$

By Theorem 1.2(iii), there is an η -quasisymmetric homeomorphism $f : (X, \hat{d}) \rightarrow \mathbb{D}^2$, where η depends only the data of X and the ratio $\text{diam } \hat{X} / \text{diam } \partial X$. Since this ratio is bounded above and below, η depends only on the data of X . Moreover, f defines a quasisymmetric homeomorphism from ∂X to $\partial \mathbb{D}^2$. We may choose a Möbius homeomorphism $g : \mathbb{D}^2 \rightarrow \mathbb{R}_+^2$ such that sequences converging to $f(\infty) \in \partial \mathbb{D}^2$ are mapped to unbounded sequences in \mathbb{R}_+^2 . Then the composition $g \circ f \circ \iota$ maps unbounded sequences to unbounded sequences. As in the complete case, it follows that $g \circ f \circ \iota$ is η' -quasisymmetric, where η' depends only on η .

Acknowledgements. The author extends heartfelt thanks to his advisor Mario Bonk for his mentorship and many of the ideas in this paper. Also, thanks to Juha Heinonen for many useful discussions.

References

1. L. AHLFORS, ‘Quasiconformal reflections’, *Acta Math.* 109 (1963) 291–301.
2. Z. BALOGH and S. BUCKLEY, ‘Sphericalization and flattening’, *Conform. Geom. Dyn.* 9 (2005) 76–101.

3. M. BONK, J. HEINONEN and E. SAKSMAN, 'The quasiconformal Jacobian problem', *Contemp. Math.* 355 (2004) 77–96.
4. M. BONK and B. KLEINER, 'Quasisymmetric parameterizations of two-dimensional metric spheres', *Invent. Math.* 150 (2002) 127–183.
5. M. BONK and B. KLEINER, 'Rigidity for quasi-Möbius group actions', *J. Differential Geom.* 61 (2002) 81–106.
6. M. BONK and B. KLEINER, 'Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary', *Geom. Topol.* 9 (2005) 219–246.
7. G. DAVID and S. SEMMES, 'Strong A_∞ -weights, Sobolev inequalities, and quasiconformal mappings', *Analysis and partial differential equations* (ed. C. Sadosky), Lecture Notes in Pure and Applied Mathematics 122 (Dekker, New York, 1990) 101–111.
8. H. FEDERER, *Geometric measure theory* (Springer, Heidelberg, 1969).
9. F. W. GEHRING, 'Univalent functions and the Schwarzian derivative', *Comment. Math. Helv.* 52 (1977) 561–572.
10. Z. HE and O. SCHRAMM, 'Fixed points, Koebe uniformization and circle packings', *Ann. Math.* 137 (1993) 369–406.
11. J. HEINONEN, *Lectures on analysis on metric spaces* (Springer, New York, 2001).
12. J. HEINONEN, 'The branch set of a quasiregular mapping', *Proceedings of the ICM*, Vol. II, Beijing, 2002, (ed. L. Tatsien; Higher Education Press, Beijing, 2002) 691–700.
13. J. MUNKRES, *Topology: a first course* (Prentice-Hall, Englewood Cliffs, NJ, 1975).
14. M. H. A. NEWMAN, *Elements of the topology of plane sets of points* (Cambridge University Press, Cambridge, 1961).
15. S. SEMMES, 'Good metric spaces without good parameterizations', *Rev. Iberoamericana* 12 (1996) 187–275.
16. S. SEMMES, *Some novel types of fractal geometry*, Oxford Mathematical Monographs (Oxford University Press, New York, 2001).
17. D. SULLIVAN, 'On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions', *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference*, State University of New York, Stony Brook, NY (Princeton University Press, Princeton, NJ, 1981) 465–496.
18. P. TUKIA, 'On quasiconformal groups', *J. Analyse Math.* 46 (1986) 318–346.
19. P. TUKIA and J. VÄISÄLÄ, 'Quasisymmetric embeddings of metric spaces', *Ann. Acad. Sci. Fenn. Ser. A I Math.* 5 (1980) 97–114.
20. J. VÄISÄLÄ, 'Quasimöbius maps', *J. Analyse Math.* 44 (1984/1985) 218–234.
21. R. L. WILDER, 'Concerning simple continuous curves and related point sets', *Amer. J. Math.* 53 (1931) 39–55.
22. K. WILDRICK, 'Quasisymmetric structures on surfaces', Preprint, 2007, arXiv:0709.0795v1 [math.MG].

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